

SUPERCONGRUENCES FOR TRUNCATED HYPERGEOMETRIC SERIES AND p -ADIC GAMMA FUNCTION

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ABSTRACT. We prove several supercongruences for truncated hypergeometric series and p -adic Gamma function. Recently, A. Deines, J. Fuselier, L. Long, H. Swisher and F. Tu posed some open problems on supercongruences based on numeric observations. We prove one of those supercongruences. Our results also generalize certain supercongruences proved by them. One of our supercongruence for ${}_7F_6$ truncated hypergeometric series is similar to a supercongruence proved by L. Long and R. Ramakrishna. A supercongruence conjectured by Rodriguez-Villegas and proved by E. Mortenson using the theory of Gaussian hypergeometric series follows from one of our more general supercongruence.

1. INTRODUCTION AND STATEMENT OF RESULTS

For a complex number a , the rising factorial or the Pochhammer symbol is defined as $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$, $k \geq 1$. If $\Gamma(x)$ denotes the Gamma function, then we have $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$. For a non-negative integer r , and $a_i, b_i \in \mathbb{C}$ with $b_i \notin \{\dots, -3, -2, -1\}$, the (generalized) hypergeometric series ${}_{r+1}F_r$ is defined by

$$(1.1) \quad {}_{r+1}F_r \left[\begin{matrix} a_1, & a_2, & \dots, & a_{r+1} \\ b_1, & \dots, & b_r \end{matrix} ; \lambda \right] := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{r+1})_k}{(b_1)_k \cdots (b_r)_k} \cdot \frac{\lambda^k}{k!},$$

which converges for $|\lambda| < 1$.

When we truncate the above sum at $k = n$, it is known as truncated hypergeometric series. We use the subscript notation to denote truncated hypergeometric series

$${}_{r+1}F_r \left[\begin{matrix} a_1, & a_2, & \dots, & a_{r+1} \\ b_1, & \dots, & b_r \end{matrix} ; \lambda \right]_n := \sum_{k=0}^n \frac{(a_1)_k \cdots (a_{r+1})_k}{(b_1)_k \cdots (b_r)_k} \cdot \frac{\lambda^k}{k!}.$$

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If one of the a_i 's is a negative integer, then hypergeometric series (1.1) terminates. More details on hypergeometric series can be found in the books by Bailey [4], Slater [25] and Andrews, Askey and Roy [3].

Hypergeometric series are of fundamental importance in many research areas including algebraic varieties, differential equations and modular forms. For instance, periods of abelian varieties such as elliptic curves, certain K3 surfaces and other Calabi-Yau manifolds can be expressed in terms of hypergeometric series. Consider the Legendre family of elliptic curves $E_\lambda : y^2 = x(x-1)(x-\lambda)$ parameterized by λ . A period of E_λ can be expressed in terms of ${}_2F_1 \left[\begin{smallmatrix} \frac{1}{2}, & \frac{1}{2} \\ 1 \end{smallmatrix}; \lambda \right]$. In [5], the first author with G. Kalita defined a period analogue for the algebraic curve $y^\ell = x(x-1)(x-\lambda)$, $\ell \geq 2$; and described it by a ${}_2F_1$ -hypergeometric series. In [19], McCarthy expressed the real period of the elliptic curve $y^2 = (x-1)(x^2 + \lambda)$ by a ${}_3F_2$ -hypergeometric series. In general, periods are complicated transcendental numbers. They are much predictable when the elliptic curve has complex multiplication (CM). The Selberg-Chowla formula predicts that any period of a CM elliptic curve is an algebraic multiple of a quotient of Gamma values [24]. In recent years, there have been focuses on a p -adic analog of these complex periods computed from the hypergeometric series and Gamma function. This often leads to congruences between truncated hypergeometric series and p -adic Gamma function modulo p . In case of CM elliptic curves, stronger congruences have been observed. These congruences are often called as *supercongruences*. These congruences are stronger than those predicted by commutative formal group theory.

Various supercongruences have been conjectured by many mathematicians including Beukers [7], van Hamme [13], Rodriguez-Villegas [29], Zudilin [30], Chan et al. [8], and many more by Z.-W. Sun [26, 27]. Some of these conjectures are proved using a variety of methods, including Gaussian hypergeometric series [1, 2, 10, 15, 20, 21, 22], the Wilf-Zeilberger method [30] and p -adic analysis [6, 23, 28]. In [17], L. Long prove several supercongruences related to special valuations of truncated hypergeometric series including a conjecture of van Hamme using hypergeometric evaluation identities and combinatorial techniques. Recently, L. Long and R. Ramakrishna [18] prove several supercongruences using a technique which relies on the relations between the classical and p -adic Gamma functions. They also prove a conjecture of Kibelek [14] and a strengthened version of a conjecture of van Hamme. For instance, they prove the following supercongruence modulo p^6 [18, Theorem 2] which is stronger than a prediction of van Hamme in [13].

(1.2)

$${}_7F_6 \left[\begin{smallmatrix} \frac{7}{6}, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3} \\ \frac{6}{6}, & 1, & 1, & 1, & 1, & 1, & 1 \end{smallmatrix}; 1 \right]_{p-1} \equiv \begin{cases} -p\Gamma_p \left(\frac{1}{3} \right)^9 & \text{if } p \equiv 1 \pmod{6}; \\ -\frac{10}{27}p^4\Gamma_p \left(\frac{1}{3} \right)^9 & \text{if } p \equiv 5 \pmod{6}, \end{cases}$$

where $\Gamma_p(\cdot)$ denotes the p -adic Gamma function recalled in Section 2.

In [9], A. Deines et al. investigate the relationships among hypergeometric series, truncated hypergeometric series, and Gaussian hypergeometric functions through the following families of algebraic varieties

$$C_{n,\lambda} : y^n = (x_1 x_2 \cdots x_{n-1})^{n-1} (1 - x_1) \cdots (1 - x_{n-1}) (x_1 - \lambda x_2 x_3 \cdots x_{n-1})$$

that are higher dimensional analogues of Legendre curves. They use information from truncated hypergeometric series to obtain information about the Galois representation and hence local zeta functions of $C_{n,\lambda}$. Motivated by certain congruences between Gaussian hypergeometric series and truncated hypergeometric series, they propose seven supercongruences for truncated hypergeometric series and p -adic Gamma function based on numeric observations. For instance, they propose a supercongruence [9, Eqn. (7.4)] considering the curve $C_{5,1}$. In this paper, we prove that their observation is correct in the following theorem.

Theorem 1.1. *Let p be a prime such that $p \equiv 1 \pmod{5}$. Then*

$${}_5F_4 \left[\begin{matrix} \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5} \\ 1, & 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1} \equiv -\Gamma_p \left(\frac{1}{5} \right)^5 \Gamma_p \left(\frac{2}{5} \right)^5 \pmod{p^5}.$$

One of the main results of A. Deines et al. is the following supercongruence [9, Theorem 7] which is true for $p \equiv 1 \pmod{4}$.

$$(1.3) \quad {}_4F_3 \left[\begin{matrix} \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4} \\ 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1} \equiv (-1)^{\frac{p-1}{4}} \Gamma_p \left(\frac{1}{2} \right) \Gamma_p \left(\frac{1}{4} \right)^6 \pmod{p^4}.$$

The following two supercongruences which we prove in this paper generalize (1.3). Though the truncated hypergeometric series present in Theorem 1.2 and Theorem 1.3 are same, but due to the presence of p on the right side of Theorem 1.3 same technique couldn't be used to prove both the theorems.

Theorem 1.2. *Let $p \geq 7$ be a prime. If $n \in \mathbb{Z}^+$ satisfies $p \equiv 1 \pmod{2n}$ then we have*

$$\begin{aligned} & {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{1}{\frac{p}{2}}, & \frac{1}{n}, & \frac{1}{n}, & \frac{2n-3}{2n} \\ \frac{5}{2n}, & 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1} \\ & \equiv (-1)^{1+\frac{p-1}{n}} \frac{\Gamma_p \left(\frac{1}{n} \right)^2 \Gamma_p \left(\frac{1}{2n} \right)^4 \Gamma_p \left(\frac{2n-3}{2n} \right)^3 \Gamma_p \left(\frac{5}{2n} \right)}{\Gamma_p \left(\frac{2}{n} \right)} \pmod{p^4}. \end{aligned}$$

Now, if we put $n = 4$ in Theorem 1.2, we deduce (1.3) for $p \equiv 1 \pmod{8}$ using the product formula (2.5) for $m = 2$ and $x = \frac{1}{4}$. If we put $n = 2$ in Theorem 1.2, we have the following supercongruence for $p \geq 7$ and $p \equiv 1 \pmod{4}$.

$$(1.4) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{4} \\ \frac{5}{4}, & 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1} := \sum_{k=0}^{p-1} \binom{2k}{k}^4 \cdot \frac{256^{-k}}{4k+1} \equiv \frac{1}{4} \Gamma_p \left(\frac{1}{4} \right)^8 \pmod{p^4}.$$

If n is odd, the statement of Theorem 1.2 is in fact true for $p \equiv 1 \pmod{n}$. If n is even, we have the following result that includes certain primes which were missed in Theorem 1.2.

Theorem 1.3. *Let $p \geq 7$ be a prime. Let $n \in \mathbb{Z}^+$ be even such that $n \neq 4$, $p \equiv 1 \pmod{n}$ and $p \not\equiv 1 \pmod{2n}$. Then we have*

$$\begin{aligned} & {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{1}{\frac{p}{2}}, & \frac{1}{n}, & \frac{1}{n}, & \frac{2n-3}{2n} \\ \frac{5}{2n}, & 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1} \\ & \equiv (-1)^{1+\frac{p-1}{n}} \frac{p}{2n} \frac{\Gamma_p \left(\frac{1}{n} \right)^2 \Gamma_p \left(\frac{1}{2n} \right)^4 \Gamma_p \left(\frac{2n-3}{2n} \right)^3 \Gamma_p \left(\frac{5}{2n} \right)}{\Gamma_p \left(\frac{2}{n} \right)} \pmod{p^4}. \end{aligned}$$

If we put $n = 2$ in Theorem 1.3, we have the following supercongruence for $p \geq 7$ and $p \equiv 3 \pmod{4}$.

$$(1.5) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{4} \\ \frac{5}{4}, & 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1} := \sum_{k=0}^{p-1} \binom{2k}{k}^4 \cdot \frac{256^{-k}}{4k+1} \equiv \frac{p}{16} \Gamma_p \left(\frac{1}{4} \right)^8 \pmod{p^4}.$$

We also prove the following supercongruence.

Theorem 1.4. *Let $p \geq 5$ be a prime such that $p \equiv 1 \pmod{n}$. Then modulo p^4 we have*

$${}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1}{n}, & \frac{1}{n}, & \frac{1}{n} \\ \frac{1}{2n}, & 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1} \equiv (-1)^{1+\frac{p-1}{n}} p \Gamma_p \left(\frac{1}{n} \right)^2 \Gamma_p \left(\frac{n-2}{n} \right).$$

If we put $n = 2$, then for primes $p \geq 5$ we have

$$(1.6) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{5}{4} \\ \frac{1}{4}, & 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1} := \sum_{k=0}^{p-1} (4k+1) \binom{2k}{k}^4 \cdot 256^{-k} \equiv p \pmod{p^4},$$

which is similar to (1.4) and (1.5). The entries $\frac{5}{4}$ and $\frac{1}{4}$ get interchanged in the ${}_5F_4$ hypergeometric series on the left side of the identities. Supercongruence (1.6) is also obtained by L. Long [17, Theorem 1].

Similar to the supercongruence (1.2), we prove the following supercongruence for ${}_7F_6$ truncated hypergeometric series modulo p^6 .

Theorem 1.5. *Let $p \equiv 1 \pmod{8}$ be a prime. Then we have*

$${}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4} \\ \frac{1}{16}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8} \end{matrix} ; 1 \right]_{p-1} \equiv -p \Gamma_p \left(\frac{7}{8} \right)^6 \Gamma_p \left(\frac{3}{8} \right)^{10} \pmod{p^6}.$$

In [29], Rodriguez-Villegas conjectured that for any odd prime p

$$(1.7) \quad {}_2F_1 \left[\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ 1 \end{matrix} ; 1 \right]_{p-1} := \sum_{k=0}^{p-1} \binom{2k}{k}^2 \cdot 16^{-k} \equiv -\Gamma_p \left(\frac{1}{2} \right)^2 \equiv \left(\frac{-1}{p} \right) \pmod{p^2}.$$

The above congruence was proved by E. Mortenson in [21] using the theory of Gaussian hypergeometric series [12] and the properties of p -adic Gamma function. Here, we prove the following result which generalizes (1.7). We note that our method does not use Gaussian hypergeometric series.

Theorem 1.6. *Let p be an odd prime such that $p \equiv 1 \pmod{n(n-1)}$. Then modulo p^2 we have*

$${}_nF_{n-1} \left[\begin{matrix} \frac{1}{n}, & \frac{1}{n}, & \dots, & \frac{1}{n} \\ \frac{1}{n-1}, & \dots, & \frac{1}{n-1} \end{matrix} ; 1 \right]_{\frac{p-1}{n-1}} \equiv (-1)^n \Gamma_p \left(\frac{1}{n-1} \right)^{n-1} \Gamma_p \left(\frac{n-1}{n} \right)^n.$$

If we put $n = 2$ in Theorem 1.6, then (1.7) follows.

2. PRELIMINARIES

We first recall the definition of the p -adic Gamma function and list some of its main properties. For further details, see [16]. Let \mathbb{Z}_p denote the ring of p -adic integers and \mathbb{Q}_p denote the field of p -adic numbers. Let v_p and $|\cdot|_p$ denote the p -adic valuation and absolute value on \mathbb{Q}_p , respectively. The p -adic gamma function Γ_p is defined by setting $\Gamma_p(0) = 1$, and for $n \in \mathbb{Z}^+$ by

$$\Gamma_p(n) := (-1)^n \prod_{\substack{0 < j < n \\ p \nmid j}} j.$$

The function has a unique extension to a continuous function $\Gamma_p : \mathbb{Z}_p \rightarrow \mathbb{Z}_p^\times$. If $x \in \mathbb{Z}_p$ and $x \neq 0$, then $\Gamma_p(x)$ is defined as

$$\Gamma_p(x) := \lim_{x_n \rightarrow x} \Gamma_p(x_n),$$

where x_n runs through any sequence of positive integers p -adically approaching x . This function is locally analytic and has a Taylor series expansion

$$\Gamma_p(x+z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{Q}_p,$$

with radius of convergence $\varrho = p^{-\frac{1}{p} - \frac{1}{p-1}}$.

We now list some of the main properties of Γ_p in the following proposition.

Proposition 2.1. *Let $x \in \mathbb{Z}_p$. We have*

- (1) $\Gamma_p(0) = 1$ and $\Gamma_p(1) = -1$.
- (2) $\frac{\Gamma_p(1+x)}{\Gamma_p(x)} = \begin{cases} -x, & \text{if } |x|_p = 1; \\ -1, & \text{if } |x|_p < 1. \end{cases}$
- (3) $\Gamma_p(1-x)\Gamma_p(x) = (-1)^{a_0(x)}$, where $a_0(x) \in \{1, 2, \dots, p\}$ such that $x \equiv a_0(x) \pmod{p}$.
- (4) $\Gamma_p\left(\frac{1}{2}\right)^2 = (-1)^{\frac{p+1}{2}}$.

For $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$, we define

$$G_n(x) := \frac{\Gamma_p^{(n)}(x)}{\Gamma_p(x)}.$$

In particular, $G_0(x) = 1$. We now state two results about Taylor series of Γ_p from [18]. Let \mathbb{C}_p denote the completion of the algebraic closure of \mathbb{Q}_p .

Proposition 2.2. [18, Proposition 13] *Let $p \geq 5$ be a prime and $a \in \mathbb{Q}$ with $v_p(a) \geq 0$. Then $v_p\left(\frac{G_i(a)}{i!}\right) \geq -i\left(\frac{1}{p} + \frac{1}{p-1}\right)$. For $i < p$, $v_p\left(\frac{G_i(a)}{i!}\right) = 0$. We*

may extend the domain of $\Gamma_p(a+x)$ by setting $\Gamma_p(a+x) = \Gamma_p(a) \cdot \sum_{k=0}^{\infty} \frac{G_k(a)}{k!} x^k$ for $x \in \mathbb{C}_p$ with $v_p(x) \geq \left(\frac{1}{p} + \frac{1}{p-1}\right)$. In particular,

$$(2.1) \quad \frac{\Gamma_p(a+1+x)}{\Gamma_p(a+x)} = \begin{cases} -(a+x), & \text{if } |a+x|_p = 1; \\ -1, & \text{if } |a+x|_p < 1, \end{cases}$$

and $\Gamma_p(a+x)\Gamma_p(1-a-x) = (-1)^{a_0(a)}$.

Theorem 2.3. [18, Theorem 14] *For $p \geq 5, r \in \mathbb{N}, a \in \mathbb{Z}_p, m \in \mathbb{C}_p$ satisfying $v_p(m) \geq 0$ and $t \in \{0, 1, 2\}$ we have*

$$(2.2) \quad \frac{\Gamma_p(a + mp^r)}{\Gamma_p(a)} \equiv \sum_{k=0}^t \frac{G_k(a)}{k!} (mp^r)^k \pmod{p^{(1+t)r}}.$$

The above result also holds for $t = 4$ if $p \geq 11$.

Lemma 2.4. [28, Lemma 2.1] *Let p be prime and ζ a primitive n -th root of unity for some positive integer n . If $a, b \in \mathbb{Q} \cap \mathbb{Z}_p^\times$ and k is a positive integer such that $(a + j) \in \mathbb{Z}_p^\times$ for each $0 \leq j \leq k - 1$, then*

$$(a - bp)_k (a - b\zeta p)_k \cdots (a - b\zeta^{n-1} p)_k \equiv (a)_k^n \pmod{p^n}.$$

Moreover for an indeterminate x ,

$$(a - bx)_k (a - b\zeta x)_k \cdots (a - b\zeta^{n-1} x)_k \in \mathbb{Z}_p[[x^n]].$$

Let $x \in \mathbb{C}_p$ be such that $v_p(x) \geq \left(\frac{1}{p} + \frac{1}{p-1}\right)$ and $|x + j|_p = 1$ for each $0 \leq j \leq k - 1$. Then using (2.1) repeatedly, we deduce that

$$(2.3) \quad (x)_k = (-1)^k \frac{\Gamma_p(x + k)}{\Gamma_p(x)}.$$

If for $j = j_1, j_2, \dots, j_i$, $|x + j|_p < 1$ and $|x + j|_p = 1$ for all other values of j , then using (2.1) repeatedly, we deduce that

$$(2.4) \quad (x)_k = (-1)^k (x + j_1) \cdots (x + j_i) \frac{\Gamma_p(x + k)}{\Gamma_p(x)}.$$

We now state a product formula for the p -adic Gamma function. Let \mathbb{F}_p denote the finite field with p elements. Let $\omega : \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\times$ be the Teichmüller character. For $a \in \mathbb{F}_p^\times$, the value $\omega(a)$ is just the $(p - 1)$ -th root of unity in \mathbb{Z}_p such that $\omega(a) \equiv a \pmod{p}$. If $m \in \mathbb{Z}^+$, $p \nmid m$ and $x = \frac{r}{p-1}$ with $0 \leq r \leq p - 1$, then

$$(2.5) \quad \prod_{h=0}^{m-1} \Gamma_p\left(\frac{x+h}{m}\right) = \omega\left(m^{(1-x)(1-p)}\right) \Gamma_p(x) \prod_{h=1}^{m-1} \Gamma_p\left(\frac{h}{m}\right).$$

Finally, we recall some hypergeometric formulae from [3, 4, 11, 25]. We first state the Whipple's well-posed ${}_7F_6$ evaluation formula.

Theorem 2.5. [3, Theorem 3.4.5] *We have*

$$\begin{aligned} & {}_7F_6 \left[\begin{matrix} a, & \frac{a}{2} + 1, & b, & c, & d, & e, & f \\ & \frac{a}{2}, & 1 + a - b, & 1 + a - c, & 1 + a - d, & 1 + a - e, & 1 + a - f \end{matrix} ; 1 \right] \\ &= \frac{\Gamma(1 + a - d)\Gamma(1 + a - e)\Gamma(1 + a - f)\Gamma(1 + a - d - e - f)}{\Gamma(1 + a)\Gamma(1 + a - e - f)\Gamma(1 + a - d - e)\Gamma(1 + a - d - f)} \\ & \quad \times {}_4F_3 \left[\begin{matrix} 1 + a - b - c, & d, & e, & f \\ & 1 + a - b, & 1 + a - c, & d + e + f - a \end{matrix} ; 1 \right], \end{aligned}$$

provided the left side converges and the right side terminates.

We next state the Dougall's formula.

Theorem 2.6. [3, Theorem 3.5.1] *If f is a negative integer and $1 + 2a = b + c + d + e + f$ then*

$$\begin{aligned} {}_7F_6 \left[\begin{matrix} a, & \frac{a}{2} + 1, & b, & c, & d, & e, & f \\ & \frac{a}{2}, & 1 + a - b, & 1 + a - c, & 1 + a - d, & 1 + a - e, & 1 + a - f \end{matrix} ; 1 \right] \\ = \frac{(1+a)_{-f}(1+a-b-c)_{-f}(1+a-b-d)_{-f}(1+a-c-d)_{-f}}{(1+a-b)_{-f}(1+a-c)_{-f}(1+a-d)_{-f}(1+a-b-c-d)_{-f}}. \end{aligned}$$

We also need the following identity to prove our main results.

Theorem 2.7. [25, p. 56] *If m is a positive integer then*

$${}_5F_4 \left[\begin{matrix} a, & 1 + \frac{a}{2}, & b, & c, & -m \\ & \frac{a}{2}, & 1 + a - b, & 1 + a - c, & 1 + a + m \end{matrix} ; 1 \right] = \frac{(1+a)_m(1+a-b-c)_m}{(1+a-b)_m(1+a-c)_m}.$$

We now state the following particular case of the Karlson-Minton formula.

Theorem 2.8. [11, Eqn. (1.9.3)] *For non-negative integers m_1, m_2, \dots, m_n we have*

$$\begin{aligned} {}_{n+1}F_n \left[\begin{matrix} -(m_1 + m_2 + \dots + m_n), & b_1 + m_1, & \dots, & b_n + m_n \\ & b_1, & \dots, & b_n \end{matrix} ; 1 \right] \\ = (-1)^{m_1+m_2+\dots+m_n} \frac{(m_1 + m_2 + \dots + m_n)!}{(b_1)_{m_1} \dots (b_n)_{m_n}}. \end{aligned}$$

3. PROOF OF THE RESULTS

Various supercongruences have been proved by many mathematicians using a variety of methods. Recently, L. Long and R. Ramakrishna [18] and A. Deines et al. [9] prove several supercongruences using a technique which relies on the relations between the classical and p -adic Gamma functions. We use a similar technique. In the proof, we consider certain classical hypergeometric evaluation identities and then use the relation between classical and p -adic Gamma function.

Proof of Theorem 1.1. Let $a = \frac{2}{5}$, $b = \frac{2}{5}$, $c = \frac{1}{5}$, $d = \frac{2(1+up)}{5}$, $e = \frac{2(1+(1-u)p)}{5}$ and $f = \frac{2(1-p)}{5}$, where u is any p -adic integer. Substituting all these values in Theorem 2.6 we obtain

$$\begin{aligned} {}_7F_6 \left[\begin{matrix} \frac{2}{5}, & 1 + \frac{1}{5}, & \frac{2}{5}, & \frac{1}{5}, & \frac{2(1+up)}{5}, & \frac{2(1+(1-u)p)}{5}, & \frac{2(1-p)}{5} \\ & \frac{1}{5}, & 1, & 1 + \frac{1}{5}, & 1 - \frac{2up}{5}, & 1 - \frac{2(1-u)p}{5}, & 1 + \frac{2p}{5} \end{matrix} ; 1 \right]_{\frac{2(p-1)}{5}} \\ = \frac{(\frac{7}{5})_{\frac{2(p-1)}{5}} (\frac{4}{5})_{\frac{2(p-1)}{5}} (\frac{3}{5} - \frac{2up}{5})_{\frac{2(p-1)}{5}} (\frac{4-2up}{5})_{\frac{2(p-1)}{5}}}{(1)_{\frac{2(p-1)}{5}} (1 + \frac{1}{5})_{\frac{2(p-1)}{5}} (1 - \frac{2up}{5})_{\frac{2(p-1)}{5}} (\frac{2}{5} - \frac{2up}{5})_{\frac{2(p-1)}{5}}}. \end{aligned}$$

Canceling equal entries from the top and bottom rows of the hypergeometric series we have

$$\begin{aligned} (3.1) \quad {}_5F_4 \left[\begin{matrix} \frac{2}{5}, & \frac{2}{5}, & \frac{2(1+up)}{5}, & \frac{2(1+(1-u)p)}{5}, & \frac{2(1-p)}{5} \\ & 1, & 1 - \frac{2up}{5}, & 1 - \frac{2(1-u)p}{5}, & 1 + \frac{2p}{5} \end{matrix} ; 1 \right]_{\frac{2(p-1)}{5}} \\ = \frac{(\frac{7}{5})_{\frac{2(p-1)}{5}} (\frac{4}{5})_{\frac{2(p-1)}{5}} (\frac{3}{5} - \frac{2up}{5})_{\frac{2(p-1)}{5}} (\frac{4-2up}{5})_{\frac{2(p-1)}{5}}}{(1)_{\frac{2(p-1)}{5}} (1 + \frac{1}{5})_{\frac{2(p-1)}{5}} (1 - \frac{2up}{5})_{\frac{2(p-1)}{5}} (\frac{2}{5} - \frac{2up}{5})_{\frac{2(p-1)}{5}}}. \end{aligned}$$

By definition, we have

$$\begin{aligned}
 {}_5F_4 & \left[\begin{matrix} \frac{2}{5}, & \frac{2}{5}, & \frac{2(1+up)}{5}, & \frac{2(1+(1-u)p)}{5}, & \frac{2(1-p)}{5} \\ & 1, & 1 - \frac{2up}{5}, & 1 - \frac{2(1-u)p}{5}, & 1 + \frac{2p}{5} \end{matrix} ; 1 \right]_{\frac{2(p-1)}{5}} \\
 (3.2) \quad &= \sum_{k=0}^{\frac{2(p-1)}{5}} \frac{(\frac{2}{5})_k^2 (\frac{2}{5} + \frac{2up}{5})_k (\frac{2}{5} + \frac{2(1-u)p}{5})_k (\frac{2}{5} - \frac{2p}{5})_k}{(1)_k (1 - \frac{2up}{5})_k (1 - \frac{2(1-u)p}{5})_k (1 + \frac{2p}{5})_k k!}.
 \end{aligned}$$

Now,

$$\frac{(\frac{2}{5} + \frac{2up}{5})_k}{(1 - \frac{2up}{5})_k} = \prod_{j=0}^{k-1} \frac{(\frac{2}{5} + j + \frac{2up}{5})}{(1 + j - \frac{2up}{5})} = \prod_{j=0}^{k-1} \frac{(\frac{2}{5} + j)(1 + \frac{2up}{5(\frac{2}{5}+j)})}{(1 + j)(1 - \frac{2up}{5(1+j)})}.$$

We observe that in the above product the terms $\frac{2}{5} + j$ and $1 + j$ do not contain a multiple of p for $0 \leq k \leq \frac{2(p-1)}{5}$. So, we can find constants $a_{k,1}, a_{k,2}, \dots \in \mathbb{Z}_p$ such that

$$(3.3) \quad \frac{(\frac{2}{5} + \frac{2up}{5})_k}{(1 - \frac{2up}{5})_k} = \frac{(\frac{2}{5})_k}{(1)_k} \left[1 + \sum_{i \geq 1} a_{k,i} \left(\frac{2up}{5} \right)^i \right].$$

Similarly,

$$(3.4) \quad \frac{(\frac{2}{5} + \frac{2(1-u)p}{5})_k}{(1 - \frac{2(1-u)p}{5})_k} = \frac{(\frac{2}{5})_k}{(1)_k} \left[1 + \sum_{i \geq 1} a_{k,i} \left(\frac{2(1-u)p}{5} \right)^i \right]$$

and

$$(3.5) \quad \frac{(\frac{2}{5} - \frac{2p}{5})_k}{(1 + \frac{2p}{5})_k} = \frac{(\frac{2}{5})_k}{(1)_k} \left[1 + \sum_{i \geq 1} a_{k,i} \left(\frac{-2p}{5} \right)^i \right].$$

Substituting (3.3), (3.4) and (3.5) into (3.2) we obtain

$$\begin{aligned}
 {}_5F_4 & \left[\begin{matrix} \frac{2}{5}, & \frac{2}{5}, & \frac{2(1+up)}{5}, & \frac{2(1+(1-u)p)}{5}, & \frac{2(1-p)}{5} \\ & 1, & 1 - \frac{2up}{5}, & 1 - \frac{2(1-u)p}{5}, & 1 + \frac{2p}{5} \end{matrix} ; 1 \right]_{\frac{2(p-1)}{5}} \\
 &\equiv \sum_{k=0}^{\frac{2(p-1)}{5}} \frac{(\frac{2}{5})_k^5}{k!^5} \left[1 + \sum_{i=1}^4 a_{k,i} \left(\frac{2up}{5} \right)^i \right] \left[1 + \sum_{i=1}^4 a_{k,i} \left(\frac{2(1-u)p}{5} \right)^i \right] \\
 (3.6) \quad &\times \left[1 + \sum_{i=1}^4 a_{k,i} \left(\frac{-2p}{5} \right)^i \right] \pmod{p^5}.
 \end{aligned}$$

Collecting the terms on the right side of (3.6) we obtain that the coefficients of p , p^2 , p^3 and p^4 are $\frac{2a_{k,1}}{5}(u+1-u-1) = 0$, $\frac{1}{5^2}(8a_{k,2} - 4a_{k,1}^2)(1-u+u^2)$, $\frac{1}{5^3}(24a_{k,3} + 8a_{k,1}^3 - 24a_{k,1}a_{k,2})u(u-1)$, and $\frac{1}{5^4}(16a_{k,4} - 16a_{k,1}a_{k,3} + 8a_{k,2}^2)(1+(1-u)^4+u^4)$,

respectively. Thus, modulo p^5 we have

$$\begin{aligned}
 (3.7) \quad & {}_5F_4 \left[\begin{matrix} \frac{2}{5}, & \frac{2}{5}, & \frac{2(1+up)}{5}, & \frac{2(1+(1-u)p)}{5}, & \frac{2(1-p)}{5} \\ 1, & 1 - \frac{2up}{5}, & 1 - \frac{2(1-u)p}{5}, & 1 + \frac{2p}{5} \end{matrix} ; 1 \right] \\
 & \equiv \sum_{k=0}^{\frac{2(p-1)}{5}} \frac{(\frac{2}{5})_k^5}{k!^5} [1 + A_2(1-u+u^2)p^2 + A_3u(u-1)p^3 + A_4(1+(1-u)^4+u^4)p^4],
 \end{aligned}$$

where $A_2, A_3, A_4 \in \mathbb{Z}_p$ are the respective coefficients of p^2, p^3 and p^4 as obtained above.

We now observe that on the right side of (3.1) the rising factorials $(\frac{7}{5})_{\frac{2(p-1)}{5}}$ and $(\frac{6}{5})_{\frac{2(p-1)}{5}}$ contain factors $(\frac{7}{5} + \frac{2(p-1)}{5} - 1) = \frac{2p}{5}$ and $(\frac{6}{5} + \frac{p-1}{5} - 1) = \frac{p}{5}$, respectively. So, if we use (2.3) and (2.4), and then 2nd part of Proposition 2.1, the right side of (3.1) reduces to

$$\begin{aligned}
 (3.8) \quad & \frac{(\frac{7}{5})_{\frac{2(p-1)}{5}} (\frac{4}{5})_{\frac{2(p-1)}{5}} (\frac{3}{5} - \frac{2up}{5})_{\frac{2(p-1)}{5}} (\frac{4-2up}{5})_{\frac{2(p-1)}{5}}}{(1)_{\frac{2(p-1)}{5}} (1 + \frac{1}{5})_{\frac{2(p-1)}{5}} (1 - \frac{2up}{5})_{\frac{2(p-1)}{5}} (\frac{2}{5} - \frac{2up}{5})_{\frac{2(p-1)}{5}}} \\
 & = \frac{\frac{2p}{5} \Gamma_p(1 + \frac{2p}{5}) \Gamma_p(\frac{2}{5} + \frac{2p}{5}) \Gamma_p(\frac{1}{5} + \frac{2(1-u)p}{5}) \Gamma_p(\frac{2}{5} + \frac{2(1-u)p}{5})}{\Gamma_p(1 + \frac{2}{5}) \Gamma_p(\frac{4}{5}) \Gamma_p(\frac{3}{5} - \frac{2up}{5}) \Gamma_p(\frac{4}{5} - \frac{2up}{5})} \\
 & \times \frac{\Gamma_p(1) \Gamma_p(1 + \frac{1}{5}) \Gamma_p(1 - \frac{2up}{5}) \Gamma_p(\frac{2}{5} - \frac{2up}{5})}{\Gamma_p(\frac{3}{5} + \frac{2p}{5}) \Gamma_p(\frac{4}{5} + \frac{2p}{5}) \Gamma_p(\frac{3}{5} + \frac{2(1-u)p}{5}) \Gamma_p(\frac{2(1-u)p}{5})} \\
 & = \frac{-\Gamma_p(1 + \frac{2p}{5}) \Gamma_p(\frac{2}{5} + \frac{2p}{5}) \Gamma_p(\frac{1}{5} + \frac{2(1-u)p}{5}) \Gamma_p(\frac{2}{5} + \frac{2(1-u)p}{5})}{\Gamma_p(\frac{2}{5}) \Gamma_p(\frac{4}{5}) \Gamma_p(\frac{3}{5} - \frac{2up}{5}) \Gamma_p(\frac{4}{5} - \frac{2up}{5})} \\
 & \times \frac{\Gamma_p(\frac{1}{5}) \Gamma_p(1 - \frac{2up}{5}) \Gamma_p(\frac{2}{5} - \frac{2up}{5})}{\Gamma_p(\frac{3}{5} + \frac{2p}{5}) \Gamma_p(\frac{4}{5} + \frac{2p}{5}) \Gamma_p(\frac{3}{5} + \frac{2(1-u)p}{5}) \Gamma_p(\frac{2(1-u)p}{5})}.
 \end{aligned}$$

Rearranging the terms on the right side of (3.8) we have

$$\begin{aligned}
 (3.9) \quad & \frac{(\frac{7}{5})_{\frac{2(p-1)}{5}} (\frac{4}{5})_{\frac{2(p-1)}{5}} (\frac{3}{5} - \frac{2up}{5})_{\frac{2(p-1)}{5}} (\frac{4-2up}{5})_{\frac{2(p-1)}{5}}}{(1)_{\frac{2(p-1)}{5}} (1 + \frac{1}{5})_{\frac{2(p-1)}{5}} (1 - \frac{2up}{5})_{\frac{2(p-1)}{5}} (\frac{2}{5} - \frac{2up}{5})_{\frac{2(p-1)}{5}}} \\
 & = \frac{-\Gamma_p(\frac{1}{5})}{\Gamma_p(\frac{2}{5}) \Gamma_p(\frac{4}{5})} \times \frac{\Gamma_p(1 + \frac{2p}{5}) \Gamma_p(1 - \frac{2up}{5})}{\Gamma_p(\frac{2(1-u)p}{5})} \times \frac{\Gamma_p(\frac{2}{5} + \frac{2p}{5}) \Gamma_p(\frac{2}{5} - \frac{2up}{5})}{\Gamma_p(\frac{3}{5} + \frac{2(1-u)p}{5})} \\
 & \times \frac{\Gamma_p(\frac{2}{5} + \frac{2(1-u)p}{5})}{\Gamma_p(\frac{3}{5} - \frac{2up}{5}) \Gamma_p(\frac{3}{5} + \frac{2p}{5})} \times \frac{\Gamma_p(\frac{1}{5} + \frac{2(1-u)p}{5})}{\Gamma_p(\frac{4}{5} - \frac{2up}{5}) \Gamma_p(\frac{4}{5} + \frac{2p}{5})}.
 \end{aligned}$$

From Proposition 2.1 we have

$$\begin{aligned}
 & \frac{\Gamma_p(1 + \frac{2p}{5}) \Gamma_p(1 - \frac{2up}{5})}{\Gamma_p(\frac{2(1-u)p}{5})} \\
 & = -\Gamma_p\left(1 + \frac{2p}{5}\right) \Gamma_p\left(1 - \frac{2up}{5}\right) \Gamma_p\left(1 - \frac{2(1-u)p}{5}\right).
 \end{aligned}$$

Using Theorem 2.3 we obtain

$$\begin{aligned}
& \frac{\Gamma_p(1 + \frac{2p}{5})\Gamma_p(1 - \frac{2up}{5})}{\Gamma_p(\frac{2(1-u)p}{5})} \\
& \equiv -\Gamma_p(1)^3 \left[1 + \sum_{i=1}^4 \frac{G_i(1)}{i!} \left(\frac{2p}{5} \right)^i \right] \left[1 + \sum_{i=1}^4 \frac{G_i(1)}{i!} \left(\frac{-2up}{5} \right)^i \right] \\
& \times \left[1 + \sum_{i=1}^4 \frac{G_i(1)}{i!} \left(\frac{2(u-1)p}{5} \right)^i \right] \\
& \equiv 1 + \frac{1}{5^2} (4G_2(1) - 4G_1(1)^2)(1 - u + u^2)p^2 \\
& - \frac{8}{5^3} \left(G_1(1)^3 - \frac{3G_1(1)G_2(1)}{2} + \frac{G_3(1)}{2} \right) u(u-1)p^3 \\
(3.10) \quad & + \frac{1}{5^4} \left(\frac{2G_4(1)}{3} - 8 \frac{G_1(1)G_3(1)}{3} + 2G_2(1)^2 \right) (1 + u^4 + (1-u)^4)p^4 \pmod{p^5}.
\end{aligned}$$

Similarly, using Proposition 2.1 and Theorem 2.3 we have, modulo p^5

$$\begin{aligned}
& \frac{\Gamma_p(\frac{2}{5} + \frac{2p}{5})\Gamma_p(\frac{2}{5} - \frac{2up}{5})}{\Gamma_p(\frac{3}{5} + \frac{2(1-u)p}{5})} \\
& \equiv -\Gamma_p\left(\frac{2}{5}\right)^3 \left[1 + \frac{1}{5^2} \left(4G_2\left(\frac{2}{5}\right) - 4G_1\left(\frac{2}{5}\right)^2 \right) (1 - u + u^2)p^2 \right] \\
& + \Gamma_p\left(\frac{2}{5}\right)^3 \left[\frac{8}{5^3} \left(G_1\left(\frac{2}{5}\right)^3 - \frac{3G_1\left(\frac{2}{5}\right)G_2\left(\frac{2}{5}\right)}{2} + \frac{G_3\left(\frac{2}{5}\right)}{2} \right) u(u-1)p^3 \right] \\
(3.11) \quad & - \Gamma_p\left(\frac{2}{5}\right)^3 \left[\frac{1}{5^4} \left(\frac{2G_4\left(\frac{2}{5}\right)}{3} - 8 \frac{G_1\left(\frac{2}{5}\right)G_3\left(\frac{2}{5}\right)}{3} + 2G_2\left(\frac{2}{5}\right)^2 \right) (1 + u^4 + (1-u)^4)p^4 \right],
\end{aligned}$$

$$\begin{aligned}
& \frac{\Gamma_p(\frac{2}{5} + \frac{2(1-u)p}{5})}{\Gamma_p(\frac{3}{5} - \frac{2up}{5})\Gamma_p(\frac{3}{5} + \frac{2p}{5})} \\
& \equiv \Gamma_p\left(\frac{2}{5}\right)^3 \left[1 + \frac{1}{5^2} \left(4G_2\left(\frac{2}{5}\right) - 4G_1\left(\frac{2}{5}\right)^2 \right) (1 - u + u^2)p^2 \right] \\
& - \Gamma_p\left(\frac{2}{5}\right)^3 \left[\frac{8}{5^3} \left(G_1\left(\frac{2}{5}\right)^3 - \frac{3G_1\left(\frac{2}{5}\right)G_2\left(\frac{2}{5}\right)}{2} + \frac{G_3\left(\frac{2}{5}\right)}{2} \right) u(u-1)p^3 \right] \\
(3.12) \quad & + \Gamma_p\left(\frac{2}{5}\right)^3 \left[\frac{1}{5^4} \left(\frac{2G_4\left(\frac{2}{5}\right)}{3} - 8 \frac{G_1\left(\frac{2}{5}\right)G_3\left(\frac{2}{5}\right)}{3} + 2G_2\left(\frac{2}{5}\right)^2 \right) (1 + u^4 + (1-u)^4)p^4 \right]
\end{aligned}$$

and

$$\begin{aligned}
 & \frac{\Gamma_p\left(\frac{1}{5} + \frac{2(1-u)p}{5}\right)}{\Gamma_p\left(\frac{4}{5} - \frac{2up}{5}\right)\Gamma_p\left(\frac{4}{5} + \frac{2p}{5}\right)} \\
 & \equiv \Gamma_p\left(\frac{1}{5}\right)^3 \left[1 + \frac{1}{5^2} \left(4G_2\left(\frac{1}{5}\right) - 4G_1\left(\frac{1}{5}\right)^2 \right) (1-u+u^2)p^2 \right] \\
 & - \Gamma_p\left(\frac{1}{5}\right)^3 \left[\frac{8}{5^3} \left(G_1\left(\frac{1}{5}\right)^3 - \frac{3G_1\left(\frac{1}{5}\right)G_2\left(\frac{1}{5}\right)}{2} + \frac{G_3\left(\frac{1}{5}\right)}{2} \right) u(u-1)p^3 \right] \\
 (3.13) \quad & + \Gamma_p\left(\frac{1}{5}\right)^3 \left[\frac{1}{5^4} \left(\frac{2G_4\left(\frac{1}{5}\right)}{3} - 8\frac{G_1\left(\frac{1}{5}\right)G_3\left(\frac{1}{5}\right)}{3} + 2G_2\left(\frac{1}{5}\right)^2 \right) (1+u^4+(1-u)^4)p^4 \right].
 \end{aligned}$$

Substituting (3.10), (3.11), (3.12) and (3.13) into (3.9) we deduce that

$$\begin{aligned}
 & \frac{\left(\frac{7}{5}\right)^{\frac{2(p-1)}{5}} \left(\frac{4}{5}\right)^{\frac{2(p-1)}{5}} \left(\frac{3}{5} - \frac{2up}{5}\right)^{\frac{2(p-1)}{5}} \left(\frac{4-2up}{5}\right)^{\frac{2(p-1)}{5}}}{(1)^{\frac{2(p-1)}{5}} \left(1 + \frac{1}{5}\right)^{\frac{2(p-1)}{5}} \left(1 - \frac{2up}{5}\right)^{\frac{2(p-1)}{5}} \left(\frac{2}{5} - \frac{2up}{5}\right)^{\frac{2(p-1)}{5}}} \\
 & \equiv \frac{\Gamma_p\left(\frac{1}{5}\right)}{\Gamma_p\left(\frac{2}{5}\right)\Gamma_p\left(\frac{4}{5}\right)} \times \Gamma_p\left(\frac{2}{5}\right)^6 \Gamma_p\left(\frac{1}{5}\right)^3 \\
 & \times [1 + B_2(1-u+u^2)p^2 + B_3u(u-1)p^3 + B_4(1+(1-u)^4+u^4)p^4] \pmod{p^5}
 \end{aligned}$$

for some $B_2, B_3, B_4 \in \mathbb{Z}_p$. Again, using proposition 2.1 we have

$$\begin{aligned}
 & \frac{\left(\frac{7}{5}\right)^{\frac{2(p-1)}{5}} \left(\frac{4}{5}\right)^{\frac{2(p-1)}{5}} \left(\frac{3}{5} - \frac{2up}{5}\right)^{\frac{2(p-1)}{5}} \left(\frac{4-2up}{5}\right)^{\frac{2(p-1)}{5}}}{(1)^{\frac{2(p-1)}{5}} \left(1 + \frac{1}{5}\right)^{\frac{2(p-1)}{5}} \left(1 - \frac{2up}{5}\right)^{\frac{2(p-1)}{5}} \left(\frac{2}{5} - \frac{2up}{5}\right)^{\frac{2(p-1)}{5}}} \\
 & \equiv -[1 + B_2(1-u+u^2)p^2 + B_3u(u-1)p^3 + B_4(1+(1-u)^4+u^4)p^4] \\
 (3.14) \quad & \times \Gamma_p\left(\frac{2}{5}\right)^5 \Gamma_p\left(\frac{1}{5}\right)^5 \pmod{p^5}.
 \end{aligned}$$

Combining (3.7) and (3.14) we obtain

$$\begin{aligned}
 & \sum_{k=0}^{\frac{2(p-1)}{5}} \frac{\left(\frac{2}{5}\right)_k^5}{k!^5} [1 + A_2(1-u+u^2)p^2 + A_3u(u-1)p^3 + A_4(1+(1-u)^4+u^4)p^4] \\
 & \equiv -[1 + B_2(1-u+u^2)p^2 + B_3u(u-1)p^3 + B_4(1+(1-u)^4+u^4)p^4] \\
 & \times \Gamma_p\left(\frac{2}{5}\right)^5 \Gamma_p\left(\frac{1}{5}\right)^5 \pmod{p^5}
 \end{aligned}$$

for all $u \in \mathbb{Z}_p$. We now take four different values of u to form a 4×4 invertible matrix and solve the above system of equations to obtain

$$(3.15) \quad \sum_{k=0}^{\frac{2(p-1)}{5}} \frac{\left(\frac{2}{5}\right)_k^5}{k!^5} \equiv -\Gamma_p\left(\frac{2}{5}\right)^5 \Gamma_p\left(\frac{1}{5}\right)^5 \pmod{p^5}.$$

By definition, we have

$$(3.16) \quad {}_5F_4 \left[\begin{matrix} \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5} \\ 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{\frac{2(p-1)}{5}} = \sum_{k=0}^{\frac{2(p-1)}{5}} \frac{(\frac{2}{5})_k^5}{k!^5}.$$

Finally, combining (3.15) and (3.16), and using the fact that modulo p^5

$${}_5F_4 \left[\begin{matrix} \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5} \\ 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{\frac{2(p-1)}{5}} \equiv {}_5F_4 \left[\begin{matrix} \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5}, & \frac{2}{5} \\ 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1}$$

we deduce the required result. This completes the proof of the theorem. \square

Proof of Theorem 1.2. Let $a = \frac{1}{n}$, $b = \frac{2n-3}{2n}$, $c = \frac{1}{2n}$, $d = \frac{1+up}{n}$, $e = \frac{1+(1-u)p}{n}$ and $f = \frac{1-p}{n}$; where u is any p -adic integer. Plugging these values in Theorem 2.6 we have

$$(3.17) \quad {}_7F_6 \left[\begin{matrix} \frac{1}{n}, & \frac{2n+1}{2n}, & \frac{2n-3}{2n}, & \frac{1}{2n}, & \frac{1+up}{n}, & \frac{1+(1-u)p}{n}, & \frac{1-p}{n} \\ \frac{1}{2n}, & \frac{5}{2n}, & \frac{2n+1}{2n}, & 1 - \frac{up}{n}, & 1 - \frac{(1-u)p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ = \frac{(\frac{n+1}{n})_{\frac{p-1}{n}} (\frac{2}{n})_{\frac{p-1}{n}} (\frac{3}{2n} - \frac{up}{n})_{\frac{p-1}{n}} (\frac{2n-1}{2n} - \frac{up}{n})_{\frac{p-1}{n}}}{(\frac{5}{2n})_{\frac{p-1}{n}} (\frac{2n+1}{2n})_{\frac{p-1}{n}} (1 - \frac{up}{n})_{\frac{p-1}{n}} (\frac{1}{n} - \frac{up}{n})_{\frac{p-1}{n}}}.$$

Canceling the equal entries from top and bottom rows of the hypergeometric series on the left hand side of (3.17) we obtain

$$(3.18) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1+up}{n}, & \frac{1+(1-u)p}{n}, & \frac{1-p}{n} \\ \frac{5}{2n}, & 1 - \frac{up}{n}, & 1 - \frac{(1-u)p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ = \frac{(\frac{n+1}{n})_{\frac{p-1}{n}} (\frac{2}{n})_{\frac{p-1}{n}} (\frac{3}{2n} - \frac{up}{n})_{\frac{p-1}{n}} (\frac{2n-1}{2n} - \frac{up}{n})_{\frac{p-1}{n}}}{(\frac{5}{2n})_{\frac{p-1}{n}} (\frac{2n+1}{2n})_{\frac{p-1}{n}} (1 - \frac{up}{n})_{\frac{p-1}{n}} (\frac{1}{n} - \frac{up}{n})_{\frac{p-1}{n}}}.$$

From the definition of truncated hypergeometric series we have

$$(3.19) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1+up}{n}, & \frac{1+(1-u)p}{n}, & \frac{1-p}{n} \\ \frac{5}{2n}, & 1 - \frac{up}{n}, & 1 - \frac{(1-u)p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ = \sum_{k=0}^{\frac{p-1}{n}} \frac{(\frac{1}{n})_k (\frac{2n-3}{2n})_k (\frac{1+up}{n})_k (\frac{1+(1-u)p}{n})_k (\frac{1-p}{n})_k}{(\frac{5}{2n})_k (1 - \frac{up}{n})_k (1 - \frac{(1-u)p}{n})_k (1 + \frac{p}{n})_k} \frac{1}{k!}.$$

Now,

$$\frac{(\frac{1}{n} + \frac{up}{n})_k}{(1 - \frac{up}{n})_k} = \prod_{j=0}^{k-1} \frac{(\frac{1}{n} + j + \frac{up}{n})}{(1 + j - \frac{up}{n})} = \prod_{j=0}^{k-1} \frac{(\frac{1}{n} + j)(1 + \frac{up}{n(\frac{1}{n} + j)})}{(1 + j)(1 - \frac{up}{n(1+j)})}.$$

We observe that the terms $\frac{1}{n} + j$ and $1 + j$ do not contain a multiple of p for $0 \leq k \leq \frac{p-1}{n}$. Thus, there exist constants $a_{k,1}, a_{k,2}, \dots \in \mathbb{Z}_p$ such that

$$(3.20) \quad \frac{(\frac{1}{n} + \frac{up}{n})_k}{(1 - \frac{up}{n})_k} \equiv \frac{(\frac{1}{n})_k}{(1)_k} \left[1 + a_{k,1} \left(\frac{up}{n} \right) + a_{k,2} \left(\frac{up}{n} \right)^2 + a_{k,3} \left(\frac{up}{n} \right)^3 \right] \pmod{p^4}.$$

Similarly, modulo p^4 we have

$$(3.21) \quad \frac{\left(\frac{1}{n} + \frac{(1-u)p}{n}\right)_k}{\left(1 - \frac{(1-u)p}{n}\right)_k} \equiv \frac{\left(\frac{1}{n}\right)_k}{\left(1\right)_k} \left[1 + a_{k,1} \left(\frac{(1-u)p}{n}\right) + a_{k,2} \left(\frac{(1-u)p}{n}\right)^2 + a_{k,3} \left(\frac{(1-u)p}{n}\right)^3 \right]$$

and

$$(3.22) \quad \frac{\left(\frac{1}{n} - \frac{p}{n}\right)_k}{\left(1 + \frac{p}{n}\right)_k} \equiv \frac{\left(\frac{1}{n}\right)_k}{\left(1\right)_k} \left[1 + a_{k,1} \left(\frac{-p}{n}\right) + a_{k,2} \left(\frac{-p}{n}\right)^2 + a_{k,3} \left(\frac{-p}{n}\right)^3 \right].$$

Substituting (3.20), (3.21) and (3.22) into (3.19) we deduce that, for $p > 5$

$$(3.23) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1+up}{n}, & \frac{1+(1-u)p}{n}, & \frac{1-p}{n} \\ \frac{5}{2n}, & 1 - \frac{up}{n}, & 1 - \frac{(1-u)p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ \equiv \sum_{k=0}^{\frac{p-1}{n}} \frac{\left(\frac{1}{n}\right)_k^4 \left(\frac{2n-3}{2n}\right)_k}{\left(\frac{5}{2n}\right)_k (1)_k^3 k!} \left[1 + a_{k,1} \left(\frac{up}{n}\right) + a_{k,2} \left(\frac{up}{n}\right)^2 + a_{k,3} \left(\frac{up}{n}\right)^3 \right] \\ \times \left[1 + a_{k,1} \left(\frac{(1-u)p}{n}\right) + a_{k,2} \left(\frac{(1-u)p}{n}\right)^2 + a_{k,3} \left(\frac{(1-u)p}{n}\right)^3 \right] \\ \times \left[1 + a_{k,1} \left(\frac{-p}{n}\right) + a_{k,2} \left(\frac{-p}{n}\right)^2 + a_{k,3} \left(\frac{-p}{n}\right)^3 \right] \pmod{p^4}.$$

Now, coefficients of p , p^2 and p^3 on the right hand side of (3.23) are $\frac{a_{k,1}}{n}(u + (1-u) - 1) = 0$, $\frac{1}{n^2}(2a_{k,2} - a_{k,1}^2)(1 - u + u^2)$ and $\frac{1}{n^3}(a_{k,1}^3 - 3a_{k,1}a_{k,2} + 3a_{k,3})u(u-1)$, respectively. Thus, for $p > 5$ we have

$$(3.24) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1+up}{n}, & \frac{1+(1-u)p}{n}, & \frac{1-p}{n} \\ \frac{5}{2n}, & 1 - \frac{up}{n}, & 1 - \frac{(1-u)p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ \equiv \sum_{k=0}^{\frac{p-1}{n}} \frac{\left(\frac{1}{n}\right)_k^4 \left(\frac{2n-3}{2n}\right)_k}{\left(\frac{5}{2n}\right)_k (1)_k^3 k!} [1 + A_1(1 - u + u^2)p^2 + A_2u(u-1)p^3] \pmod{p^4},$$

where $A_1, A_2 \in \mathbb{Z}_p$ are the respective coefficients of p^2 and p^3 as obtained above.

Again, (3.18) and (3.24) yield

$$(3.25) \quad \sum_{k=0}^{\frac{p-1}{n}} \frac{\left(\frac{1}{n}\right)_k^4 \left(\frac{2n-3}{2n}\right)_k}{\left(\frac{5}{2n}\right)_k (1)_k^3 k!} [1 + A_1(1 - u + u^2)p^2 + A_2u(u-1)p^3] \\ \equiv \frac{\left(\frac{n+1}{n}\right)_{\frac{p-1}{n}} \left(\frac{2}{n}\right)_{\frac{p-1}{n}} \left(\frac{3}{2n} - \frac{up}{n}\right)_{\frac{p-1}{n}} \left(\frac{2n-1}{2n} - \frac{up}{n}\right)_{\frac{p-1}{n}}}{\left(\frac{5}{2n}\right)_{\frac{p-1}{n}} \left(\frac{2n+1}{2n}\right)_{\frac{p-1}{n}} \left(1 - \frac{up}{n}\right)_{\frac{p-1}{n}} \left(\frac{1}{n} - \frac{up}{n}\right)_{\frac{p-1}{n}}} \pmod{p^4}.$$

We note that the rising factorial $\left(\frac{n+1}{n}\right)_{\frac{p-1}{n}}$ on the right side of (3.25) contains a factor multiple of p , namely $\left(\frac{n+1}{n} + \frac{p-1}{n} - 1\right) = \frac{p}{n}$. So, (2.4) yields

$$\left(\frac{n+1}{n}\right)_{\frac{p-1}{n}} = (-1)^{\frac{p-1}{n}} \frac{p}{n} \frac{\Gamma_p\left(\frac{n+1}{n} + \frac{p-1}{n}\right)}{\Gamma_p\left(\frac{n+1}{n}\right)} = (-1)^{\frac{p-1}{n}} \frac{p}{n} \frac{\Gamma_p\left(1 + \frac{p}{n}\right)}{\Gamma_p\left(1 + \frac{1}{n}\right)}.$$

Also, $(\frac{2n+1}{2n})_{\frac{p-1}{n}}$ contains a factor multiple of p , namely $(\frac{2n+1}{2n} + \frac{p-1}{2n} - 1) = \frac{p}{2n}$. Thus, (2.4) gives

$$\left(\frac{2n+1}{2n}\right)_{\frac{p-1}{n}} = (-1)^{\frac{p-1}{n}} \frac{p}{2n} \frac{\Gamma_p(\frac{2n-1}{2n} + \frac{p}{n})}{\Gamma_p(\frac{2n+1}{2n})}.$$

These two are the only rising factorials in (3.25) which contain a multiple of p . We apply (2.3) for rest of the rising factorials, and then we obtain

$$\begin{aligned} & \frac{(\frac{n+1}{n})_{\frac{p-1}{n}} (\frac{2}{n})_{\frac{p-1}{n}} (\frac{3}{2n} - \frac{up}{n})_{\frac{p-1}{n}} (\frac{2n-1}{2n} - \frac{up}{n})_{\frac{p-1}{n}}}{(\frac{5}{2n})_{\frac{p-1}{n}} (\frac{2n+1}{2n})_{\frac{p-1}{n}} (1 - \frac{up}{n})_{\frac{p-1}{n}} (\frac{1}{n} - \frac{up}{n})_{\frac{p-1}{n}}} \\ &= \frac{\frac{p}{n} \Gamma_p(1 + \frac{p}{n}) \Gamma_p(\frac{1}{n} + \frac{p}{n}) \Gamma_p(\frac{1}{2n} + \frac{(1-u)p}{n}) \Gamma_p(\frac{2n-3}{2n} + \frac{(1-u)p}{n})}{\Gamma_p(1 + \frac{1}{n}) \Gamma_p(\frac{2}{n}) \Gamma_p(\frac{3}{2n} - \frac{up}{n}) \Gamma_p(\frac{2n-1}{2n} - \frac{up}{n})} \\ & \times \frac{\Gamma_p(\frac{5}{2n}) \Gamma_p(\frac{2n+1}{2n}) \Gamma_p(1 - \frac{up}{n}) \Gamma_p(\frac{1}{n} - \frac{up}{n})}{\Gamma_p(\frac{3}{2n} + \frac{p}{n}) \frac{p}{2n} \Gamma_p(\frac{2n-1}{2n} + \frac{p}{n}) \Gamma_p(\frac{n-1}{n} + \frac{(1-u)p}{n}) \Gamma_p(\frac{(1-u)p}{n})} \\ &= \frac{2 \Gamma_p(1 + \frac{p}{n}) \Gamma_p(\frac{1}{n} + \frac{p}{n}) \Gamma_p(\frac{1}{2n} + \frac{(1-u)p}{n}) \Gamma_p(\frac{2n-3}{2n} + \frac{(1-u)p}{n})}{\Gamma_p(1 + \frac{1}{n}) \Gamma_p(\frac{2}{n}) \Gamma_p(\frac{3}{2n} - \frac{up}{n}) \Gamma_p(\frac{2n-1}{2n} - \frac{up}{n})} \\ & \times \frac{\Gamma_p(\frac{5}{2n}) \Gamma_p(\frac{2n+1}{2n}) \Gamma_p(1 - \frac{up}{n}) \Gamma_p(\frac{1}{n} - \frac{up}{n})}{\Gamma_p(\frac{3}{2n} + \frac{p}{n}) \Gamma_p(\frac{2n-1}{2n} + \frac{p}{n}) \Gamma_p(\frac{n-1}{n} + \frac{(1-u)p}{n}) \Gamma_p(\frac{(1-u)p}{n})} \end{aligned}$$

Rearranging the terms we have

$$\begin{aligned} & \frac{(\frac{n+1}{n})_{\frac{p-1}{n}} (\frac{2}{n})_{\frac{p-1}{n}} (\frac{3}{2n} - \frac{up}{n})_{\frac{p-1}{n}} (\frac{2n-1}{2n} - \frac{up}{n})_{\frac{p-1}{n}}}{(\frac{5}{2n})_{\frac{p-1}{n}} (\frac{2n+1}{2n})_{\frac{p-1}{n}} (1 - \frac{up}{n})_{\frac{p-1}{n}} (\frac{1}{n} - \frac{up}{n})_{\frac{p-1}{n}}} \\ &= 2 \times \frac{\Gamma_p(\frac{5}{2n}) \Gamma_p(\frac{2n+1}{2n})}{\Gamma_p(1 + \frac{1}{n}) \Gamma_p(\frac{2}{n})} \frac{\Gamma_p(1 + \frac{p}{n}) \Gamma_p(1 - \frac{up}{n})}{\Gamma_p(\frac{(1-u)p}{n})} \frac{\Gamma_p(\frac{1}{n} + \frac{p}{n}) \Gamma_p(\frac{1}{n} - \frac{up}{n})}{\Gamma_p(\frac{n-1}{n} + \frac{(1-u)p}{n})} \\ (3.26) \quad & \times \frac{\Gamma_p(\frac{1}{2n} + \frac{(1-u)p}{n})}{\Gamma_p(\frac{2n-1}{2n} - \frac{up}{n}) \Gamma_p(\frac{2n-1}{2n} + \frac{p}{n})} \frac{\Gamma_p(\frac{2n-3}{2n} + \frac{(1-u)p}{n})}{\Gamma_p(\frac{3}{2n} - \frac{up}{n}) \Gamma_p(\frac{3}{2n} + \frac{p}{n})}. \end{aligned}$$

Applying 2nd part of Proposition 2.1 we deduce that

$$(3.27) \quad \frac{\Gamma_p(\frac{5}{2n}) \Gamma_p(\frac{2n+1}{2n})}{\Gamma_p(1 + \frac{1}{n}) \Gamma_p(\frac{2}{n})} = \frac{1}{2} \times \frac{\Gamma_p(\frac{5}{2n}) \Gamma_p(\frac{1}{2n})}{\Gamma_p(\frac{1}{n}) \Gamma_p(\frac{2}{n})}.$$

Applying Proposition 2.2, and then Theorem 2.3 we deduce that

$$\begin{aligned}
 & \frac{\Gamma_p(1 + \frac{p}{n})\Gamma_p(1 - \frac{up}{n})}{\Gamma_p(\frac{(1-u)p}{n})} \\
 &= (-1)^p \Gamma_p\left(1 + \frac{p}{n}\right) \Gamma_p\left(1 - \frac{up}{n}\right) \Gamma_p\left(1 - \frac{(1-u)p}{n}\right) \\
 &= -\Gamma_p(1)^3 \left[1 + G_1(1) \left(\frac{p}{n}\right) + \frac{G_2(1)}{2} \left(\frac{p}{n}\right)^2 + \frac{G_3(1)}{6} \left(\frac{p}{n}\right)^3 \dots\right] \\
 &\quad \times \left[1 + G_1(1) \left(\frac{-up}{n}\right) + \frac{G_2(1)}{2} \left(\frac{-up}{n}\right)^2 + \frac{G_3(1)}{6} \left(\frac{-up}{n}\right)^3 \dots\right] \\
 &\quad \times \left[1 + G_1(1) \frac{(u-1)p}{n} + \frac{G_2(1)}{2} \left(\frac{(u-1)p}{n}\right)^2 + \frac{G_3(1)}{6} \left(\frac{(u-1)p}{n}\right)^3 \dots\right] \\
 &\equiv 1 + G_1(1)(1-u+u-1)\frac{p}{n} + (G_2(1) - G_1(1)^2)(1-u+u^2)\frac{p^2}{n^2} \\
 &\quad - \left(G_1(1)^3 - \frac{3G_1(1)G_2(1)}{2} + \frac{G_3(1)}{2}\right) u(u-1)\frac{p^3}{n^3} \\
 (3.28) \quad &\equiv 1 + b_1(1-u+u^2)p^2 + b_2u(u-1)p^3 \pmod{p^4},
 \end{aligned}$$

where $b_1 = \frac{G_2(1)-G_1(1)^2}{n^2}$ and $b_2 = -\frac{G_1(1)^3-\frac{3}{2}G_1(1)G_2(1)+\frac{1}{2}G_3(1)}{n^3}$ are p -adic integers. Similarly, we deduce that modulo p^4

$$\begin{aligned}
 & \frac{\Gamma_p(\frac{1}{n} + \frac{p}{n})\Gamma_p(\frac{1}{n} - \frac{up}{n})}{\Gamma_p(\frac{n-1}{n} + \frac{(1-u)p}{n})} \\
 &= (-1)^{a_0(1-\frac{1}{n})} \Gamma_p\left(\frac{1}{n} + \frac{p}{n}\right) \Gamma_p\left(\frac{1}{n} - \frac{up}{n}\right) \Gamma_p\left(\frac{1}{n} + \frac{(u-1)p}{n}\right) \\
 (3.29) \quad &\equiv (-1)^{1+\frac{p-1}{n}} \Gamma_p\left(\frac{1}{n}\right)^3 [1 + b_3(1-u+u^2)p^2 + b_4u(u-1)p^3],
 \end{aligned}$$

(3.30)

$$\frac{\Gamma_p(\frac{1}{2n} + \frac{(1-u)p}{n})}{\Gamma_p(\frac{2n-1}{2n} - \frac{up}{n})\Gamma_p(\frac{2n-1}{2n} + \frac{p}{n})} \equiv \Gamma_p\left(\frac{1}{2n}\right)^3 [1 + b_5(1-u+u^2)p^2 + b_6u(u-1)p^3],$$

and

$$\begin{aligned}
 & \frac{\Gamma_p(\frac{2n-3}{2n} + \frac{(1-u)p}{n})}{\Gamma_p(\frac{3}{2n} - \frac{up}{n})\Gamma_p(\frac{3}{2n} + \frac{p}{n})} \\
 (3.31) \quad &\equiv \Gamma_p\left(\frac{2n-3}{2n}\right)^3 [1 + b_7(1-u+u^2)p^2 + b_8u(u-1)p^3]
 \end{aligned}$$

for some $b_3, b_4, b_5, b_6, b_7, b_8 \in \mathbb{Z}_p$. Finally, using (3.27), (3.28), (3.29), (3.30) and (3.31) in (3.26) we obtain

$$(3.32) \quad \frac{\left(\frac{n+1}{n}\right)_{\frac{p-1}{n}} \left(\frac{2}{n}\right)_{\frac{p-1}{n}} \left(\frac{3}{2n} - \frac{up}{n}\right)_{\frac{p-1}{n}} \left(\frac{2n-1}{2n} - \frac{up}{n}\right)_{\frac{p-1}{n}}}{\left(\frac{5}{2n}\right)_{\frac{p-1}{n}} \left(\frac{2n+1}{2n}\right)_{\frac{p-1}{n}} \left(1 - \frac{up}{n}\right)_{\frac{p-1}{n}} \left(\frac{1}{n} - \frac{up}{n}\right)_{\frac{p-1}{n}}} \\ \equiv (-1)^{1+\frac{p-1}{n}} \frac{\Gamma_p\left(\frac{1}{n}\right)^2 \Gamma_p\left(\frac{1}{2n}\right)^4 \Gamma_p\left(\frac{2n-3}{2n}\right)^3 \Gamma_p\left(\frac{5}{2n}\right)}{\Gamma_p\left(\frac{2}{n}\right)} \\ \times [1 + B_1(1-u+u^2)p^2 + B_2u(u-1)p^3] \pmod{p^4}$$

for some $B_1, B_2 \in \mathbb{Z}_p$. Combining (3.25) and (3.32) we obtain

$$(3.33) \quad \sum_{k=0}^{\frac{p-1}{n}} \frac{\left(\frac{1}{n}\right)_k^4 \left(\frac{2n-3}{2n}\right)_k}{\left(\frac{5}{2n}\right)_k (1)_k^3 k!} [1 + A_1(1-u+u^2)p^2 + A_2u(u-1)p^3] \\ \equiv (-1)^{1+\frac{p-1}{n}} \frac{\Gamma_p\left(\frac{1}{n}\right)^2 \Gamma_p\left(\frac{1}{2n}\right)^4 \Gamma_p\left(\frac{2n-3}{2n}\right)^3 \Gamma_p\left(\frac{5}{2n}\right)}{\Gamma_p\left(\frac{2}{n}\right)} \\ \times [1 + B_1(1-u+u^2)p^2 + B_2u(u-1)p^3] \pmod{p^4}$$

for all $u \in \mathbb{Z}_p$. We consider three different values of u to form a 3×3 invertible matrix, and then solve the system of equations (3.33) to obtain

$$(3.34) \quad \sum_{k=0}^{\frac{p-1}{n}} \frac{\left(\frac{1}{n}\right)_k^4 \left(\frac{2n-3}{2n}\right)_k}{\left(\frac{5}{2n}\right)_k (1)_k^3 k!} \equiv (-1)^{1+\frac{p-1}{n}} \frac{\Gamma_p\left(\frac{1}{n}\right)^2 \Gamma_p\left(\frac{1}{2n}\right)^4 \Gamma_p\left(\frac{2n-3}{2n}\right)^3 \Gamma_p\left(\frac{5}{2n}\right)}{\Gamma_p\left(\frac{2}{n}\right)} \pmod{p^4}.$$

By definition, we have

$$(3.35) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{1}{\frac{p}{5}}, & \frac{1}{n}, & \frac{1}{n}, & \frac{2n-3}{2n} \\ & \frac{5}{2n}, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{\frac{p-1}{n}} = \sum_{k=0}^{\frac{p-1}{n}} \frac{\left(\frac{1}{n}\right)_k^4 \left(\frac{2n-3}{2n}\right)_k}{\left(\frac{5}{2n}\right)_k (1)_k^3 k!},$$

and modulo p^4 we observe that

$$(3.36) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{1}{\frac{p}{5}}, & \frac{1}{n}, & \frac{1}{n}, & \frac{2n-3}{2n} \\ & \frac{5}{2n}, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \equiv {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{1}{\frac{p}{5}}, & \frac{1}{n}, & \frac{1}{n}, & \frac{2n-3}{2n} \\ & \frac{5}{2n}, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{\frac{p-1}{n}}.$$

Now, using (3.34), (3.35) and (3.36) we obtain the desired result. \square

Proof of Theorem 1.3. Let $a = \frac{1}{n}$, $b = \frac{2n-3}{2n}$, $c = \frac{1}{2n}$, $d = \frac{1-\zeta p}{n}$, $e = \frac{1-\zeta^2 p}{n}$ and $f = \frac{1-p}{n}$, where ζ is a primitive cubic root of unity. Plugging these values in Theorem 2.6 we have

$$(3.37) \quad {}_7F_6 \left[\begin{matrix} \frac{1}{n}, & \frac{2n+1}{2n}, & \frac{2n-3}{2n}, & \frac{1}{2n}, & \frac{1-\zeta p}{n}, & \frac{1-\zeta^2 p}{n}, & \frac{1-p}{n} \\ & \frac{1}{2n}, & \frac{5}{2n}, & \frac{2n+1}{2n}, & 1 + \frac{\zeta p}{n}, & 1 + \frac{\zeta^2 p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ = {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1-\zeta p}{n}, & \frac{1-\zeta^2 p}{n}, & \frac{1-p}{n} \\ & \frac{5}{2n}, & 1 + \frac{\zeta p}{n}, & 1 + \frac{\zeta^2 p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ = \frac{\left(\frac{n+1}{n}\right)_{\frac{p-1}{n}} \left(\frac{2}{n}\right)_{\frac{p-1}{n}} \left(\frac{3}{2n} + \frac{\zeta p}{n}\right)_{\frac{p-1}{n}} \left(\frac{2n-1}{2n} + \frac{\zeta p}{n}\right)_{\frac{p-1}{n}}}{\left(\frac{5}{2n}\right)_{\frac{p-1}{n}} \left(\frac{2n+1}{2n}\right)_{\frac{p-1}{n}} \left(1 + \frac{\zeta p}{n}\right)_{\frac{p-1}{n}} \left(1 + \frac{\zeta^2 p}{n}\right)_{\frac{p-1}{n}}}.$$

We now prove that

$$\begin{aligned}
 & {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1-\zeta p}{n}, & \frac{1-\zeta^2 p}{n}, & \frac{1-p}{n} \\ & \frac{5}{2n}, & 1 + \frac{\zeta p}{n}, & 1 + \frac{\zeta^2 p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\
 & \equiv {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{1}{p}, & \frac{1}{n}, & \frac{1}{n}, & \frac{2n-3}{2n} \\ & \frac{5}{2n}, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\
 (3.38) \quad & \equiv {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{1}{p}, & \frac{1}{n}, & \frac{1}{n}, & \frac{2n-3}{2n} \\ & \frac{5}{2n}, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1} \pmod{p^4}.
 \end{aligned}$$

The last congruence follows from the fact that the rising factorial $(\frac{1}{n})_k$ contains a multiple of p for $\frac{p-1}{n} < k \leq p-1$. As shown in the proof of Theorem 1.2, we can find $a_{k,1}, a_{k,2}, \dots \in \mathbb{Z}_p$ such that

$$(3.39) \quad \frac{(\frac{1}{n} - \frac{x}{n})_k}{(1 + \frac{x}{n})_k} = \frac{(\frac{1}{n})_k}{(1)_k} \left[1 + \sum_{i \geq 1} a_{k,i} x^i \right]$$

for $0 \leq k \leq \frac{p-1}{n}$. Therefore,

$$\begin{aligned}
 & {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1-x}{n}, & \frac{1-y}{n}, & \frac{1-p}{n} \\ & \frac{5}{2n}, & 1 + \frac{x}{n}, & 1 + \frac{y}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\
 & \equiv {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1-x}{n}, & \frac{1-y}{n}, & \frac{1}{n} \\ & \frac{5}{2n}, & 1 + \frac{x}{n}, & 1 + \frac{y}{n}, & 1 \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \pmod{p} \\
 (3.40) \quad & = \sum_{k=0}^{\frac{p-1}{n}} \frac{(\frac{2n-3}{2n})_k (\frac{1}{n})_k^4}{(\frac{5}{2n})_k (1)_k^3 k!} \left[1 + \sum_{i \geq 1} a_{k,i} x^i \right] \left[1 + \sum_{i \geq 1} a_{k,i} y^i \right] \in \mathbb{Z}_p[[x, y]].
 \end{aligned}$$

If we put $a = \frac{1}{n}$, $b = \frac{2n-3}{2n}$, $c = \frac{1}{2n}$, $d = \frac{1-x}{n}$, $e = \frac{1-y}{n}$ and $f = \frac{1-p}{n}$ in Theorem 2.5 we obtain

$$\begin{aligned}
 & {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1-x}{n}, & \frac{1-y}{n}, & \frac{1-p}{n} \\ & \frac{5}{2n}, & 1 + \frac{x}{n}, & 1 + \frac{y}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\
 (3.41) \quad & = \frac{(\frac{n+1}{n})_{\frac{p-1}{n}} (\frac{n-1}{n} + \frac{x+y}{n})_{\frac{p-1}{n}}}{(1 + \frac{x}{n})_{\frac{p-1}{n}} (1 + \frac{y}{n})_{\frac{p-1}{n}}} {}_4F_3 \left[\begin{matrix} \frac{2}{n}, & \frac{1-x}{n}, & \frac{1-y}{n}, & \frac{1-p}{n} \\ & \frac{2}{n} - \frac{x+y+p}{n}, & \frac{5}{2n}, & \frac{2n+1}{2n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}}.
 \end{aligned}$$

Here, we note that the rising factorial $(\frac{n+1}{n})_{\frac{p-1}{n}}$ contains a factor multiple of p , namely $(\frac{n+1}{n} + \frac{p-1}{n} - 1) = \frac{p}{n}$; and if $n \neq 4$, $p \not\equiv 1 \pmod{2n}$, then the rising factorials $(\frac{5}{2n})_{\frac{p-1}{n}}$ and $(\frac{2n+1}{2n})_{\frac{p-1}{n}}$ do not contain a multiple of p . This implies that

$$(3.42) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1-x}{n}, & \frac{1-y}{n}, & \frac{1-p}{n} \\ & \frac{5}{2n}, & 1 + \frac{x}{n}, & 1 + \frac{y}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \in p\mathbb{Z}_p[[x, y]].$$

Now, let $x = \zeta u$ and $y = \zeta^2 u$. Then (3.40) and Lemma 2.4 yield

$$\begin{aligned}
 & {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1-\zeta u}{n}, & \frac{1-\zeta^2 u}{n}, & \frac{1-u}{n} \\ & \frac{5}{2n}, & 1 + \frac{\zeta u}{n}, & 1 + \frac{\zeta^2 u}{n}, & 1 + \frac{u}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\
 (3.43) \quad & = A_0 + \sum_{i \geq 1} A_{3i} u^{3i} \in \mathbb{Z}_p[[u^3]],
 \end{aligned}$$

where $A_0 = \sum_{k=0}^{\frac{p-1}{n}} \frac{(\frac{2n-3}{2n})_k (\frac{1}{n})_k^4}{(\frac{5}{2n})_k (1)_k^3 k!}$ and $A_3 = \sum_{k=0}^{\frac{p-1}{n}} \frac{(\frac{2n-3}{2n})_k (\frac{1}{n})_k^4}{(\frac{5}{2n})_k (1)_k^3 k!} H(a_{k,i})$ and $H(a_{k,i})$ is an integral polynomial in the $a_{k,i}$, where the second subscripts in each monomial add to 3. If we put $u = 0$ and p , then (3.42) implies that A_0 and $A_3 \in p\mathbb{Z}_p$. Thus, we obtain

$$\begin{aligned} & {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{2n-3}{2n}, & \frac{1-\zeta p}{n}, & \frac{1-\zeta^2 p}{n}, & \frac{1-p}{n} \\ & \frac{5}{2n}, & 1 + \frac{\zeta p}{n}, & 1 + \frac{\zeta^2 p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ & \equiv \sum_{k=0}^{\frac{p-1}{n}} \frac{(\frac{2n-3}{2n})_k (\frac{1}{n})_k^4}{(\frac{5}{2n})_k (1)_k^3 k!} \\ & \equiv {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{1}{\frac{p}{n}}, & \frac{1}{n}, & \frac{1}{n}, & \frac{2n-3}{2n} \\ & \frac{5}{2n}, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \pmod{p^4}, \end{aligned}$$

which completes the proof of (3.38).

Again, from the right hand side of (3.37) we have

$$\begin{aligned} & {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & \frac{1}{\frac{p}{n}}, & \frac{1}{n}, & \frac{1}{n}, & \frac{2n-3}{2n} \\ & \frac{5}{2n}, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ (3.44) \quad & \equiv \frac{(\frac{n+1}{n})_{\frac{p-1}{n}} (\frac{2}{n})_{\frac{p-1}{n}} (\frac{3}{2n} + \frac{\zeta p}{n})_{\frac{p-1}{n}} (\frac{2n-1}{2n} + \frac{\zeta p}{n})_{\frac{p-1}{n}}}{(\frac{5}{2n})_{\frac{p-1}{n}} (\frac{2n+1}{2n})_{\frac{p-1}{n}} (1 + \frac{\zeta p}{n})_{\frac{p-1}{n}} (\frac{1}{n} + \frac{\zeta p}{n})_{\frac{p-1}{n}}} \pmod{p^4}. \end{aligned}$$

Here, we observe that the rising factorial $(\frac{n+1}{n})_{\frac{p-1}{n}}$ contains a factor multiple of p , namely $(\frac{n+1}{n} + \frac{p-1}{n} - 1) = \frac{p}{n}$; and if $n \neq 4, p \not\equiv 1 \pmod{2n}$, then no other terms on the right side of (3.44) contain a multiple of p . Following similar steps as shown in the proof of Theorem 1.2, we deduce that the right side of (3.44) becomes

$$\begin{aligned} & \frac{(\frac{n+1}{n})_{\frac{p-1}{n}} (\frac{2}{n})_{\frac{p-1}{n}} (\frac{3}{2n} + \frac{\zeta p}{n})_{\frac{p-1}{n}} (\frac{2n-1}{2n} + \frac{\zeta p}{n})_{\frac{p-1}{n}}}{(\frac{5}{2n})_{\frac{p-1}{n}} (\frac{2n+1}{2n})_{\frac{p-1}{n}} (1 + \frac{\zeta p}{n})_{\frac{p-1}{n}} (\frac{1}{n} + \frac{\zeta p}{n})_{\frac{p-1}{n}}} \\ & = \frac{p}{2n} \frac{\Gamma_p(1 + \frac{p}{n}) \Gamma_p(\frac{1}{n} + \frac{p}{n}) \Gamma_p(\frac{1}{2n} + \frac{(1+\zeta)p}{n}) \Gamma_p(\frac{2n-3}{2n} + \frac{(1+\zeta)p}{n})}{\Gamma_p(\frac{1}{n}) \Gamma_p(\frac{2}{n}) \Gamma_p(\frac{3}{2n} + \frac{\zeta p}{n}) \Gamma_p(\frac{2n-1}{2n} + \frac{\zeta p}{n})} \\ (3.45) \quad & \times \frac{\Gamma_p(\frac{5}{2n}) \Gamma_p(\frac{1}{2n}) \Gamma_p(1 + \frac{\zeta p}{n}) \Gamma_p(\frac{1}{n} + \frac{\zeta p}{n})}{\Gamma_p(\frac{3}{2n} + \frac{p}{n}) \Gamma_p(\frac{2n-1}{2n} + \frac{p}{n}) \Gamma_p(\frac{n-1}{n} + \frac{(1+\zeta)p}{n}) \Gamma_p(\frac{(1+\zeta)p}{n})}. \end{aligned}$$

Now, arranging the terms with respect to the symmetry of cubic roots of unity, and then applying 2nd part of Proposition 2.2 and Theorem 2.3 we obtain

$$(3.46) \quad \frac{\Gamma_p(1 + \frac{p}{n}) \Gamma_p(1 + \frac{\zeta p}{n})}{\Gamma_p(-\frac{\zeta^2 p}{n})} = 1 + O(p^3),$$

$$\begin{aligned} & \frac{\Gamma_p(\frac{1}{n} + \frac{p}{n}) \Gamma_p(\frac{1}{n} + \frac{\zeta p}{n})}{\Gamma_p(\frac{n-1}{n} - \frac{\zeta^2 p}{n})} = (-1)^{a_0(1-\frac{1}{n})} \Gamma_p\left(\frac{1}{n} + \frac{p}{n}\right) \Gamma_p\left(\frac{1}{n} + \frac{\zeta p}{n}\right) \Gamma_p\left(\frac{1}{n} + \frac{\zeta^2 p}{n}\right) \\ (3.47) \quad & = (-1)^{1+\frac{p-1}{n}} \Gamma_p\left(\frac{1}{n}\right)^3 (1 + O(p^3)), \end{aligned}$$

$$(3.48) \quad \frac{\Gamma_p(\frac{1}{2n} - \frac{\zeta^2 p}{n})}{\Gamma_p(\frac{2n-1}{2n} + \frac{\zeta p}{n})\Gamma_p(\frac{2n-1}{2n} + \frac{p}{n})} = \Gamma_p\left(\frac{1}{2n}\right)^3 (1 + O(p^3)),$$

and

$$(3.49) \quad \frac{\Gamma_p(\frac{2n-3}{2n} - \frac{\zeta^2 p}{n})}{\Gamma_p(\frac{3}{2n} + \frac{\zeta p}{n})\Gamma_p(\frac{3}{2n} + \frac{p}{n})} = \Gamma_p\left(\frac{2n-3}{2n}\right)^3 (1 + O(p^3)).$$

Substituting (3.46), (3.47), (3.48) and (3.49) into (3.45) we conclude that

$$(3.50) \quad \frac{\left(\frac{n+1}{n}\right)^{\frac{p-1}{n}} \left(\frac{2}{n}\right)^{\frac{p-1}{n}} \left(\frac{3}{2n} + \frac{\zeta p}{n}\right)^{\frac{p-1}{n}} \left(\frac{2n-1}{2n} + \frac{\zeta p}{n}\right)^{\frac{p-1}{n}}}{\left(\frac{5}{2n}\right)^{\frac{p-1}{n}} \left(\frac{2n+1}{2n}\right)^{\frac{p-1}{n}} \left(1 + \frac{\zeta p}{n}\right)^{\frac{p-1}{n}} \left(\frac{1}{n} + \frac{\zeta p}{n}\right)^{\frac{p-1}{n}}} \\ \equiv (-1)^{1+\frac{p-1}{n}} \frac{p}{2n} \frac{\Gamma_p\left(\frac{1}{n}\right)^2 \Gamma_p\left(\frac{1}{2n}\right)^4 \Gamma_p\left(\frac{2n-3}{2n}\right)^3 \Gamma_p\left(\frac{5}{2n}\right)}{\Gamma_p\left(\frac{2}{n}\right)} \pmod{p^4}.$$

Finally, combining (3.38), (3.44) and (3.50) we complete the proof of the theorem. \square

Proof of Theorem 1.4. Let ζ be a primitive cubic root of unity. Putting $a = \frac{1}{n}$, $b = \frac{1-\zeta p}{n}$, $c = \frac{1-\zeta^2 p}{n}$ and $m = \frac{p-1}{n}$ in Theorem 2.7 we have

$$(3.51) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1-\zeta p}{n}, & \frac{1-\zeta^2 p}{n}, & \frac{1-p}{n} \\ & \frac{1}{2n}, & 1 + \frac{\zeta p}{n}, & 1 + \frac{\zeta^2 p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ = \frac{\left(1 + \frac{1}{n}\right)^{\frac{p-1}{n}} \left(\frac{n-1}{n} + \frac{(\zeta+\zeta^2)p}{n}\right)^{\frac{p-1}{n}}}{\left(1 + \frac{\zeta p}{n}\right)^{\frac{p-1}{n}} \left(1 + \frac{\zeta^2 p}{n}\right)^{\frac{p-1}{n}}}.$$

We first show that

$$(3.52) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1-\zeta p}{n}, & \frac{1-\zeta^2 p}{n}, & \frac{1-p}{n} \\ & \frac{1}{2n}, & 1 + \frac{\zeta p}{n}, & 1 + \frac{\zeta^2 p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ \equiv {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1}{n}, & \frac{1}{n}, & \frac{1}{n} \\ & \frac{1}{2n}, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\ \equiv {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1}{n}, & \frac{1}{n}, & \frac{1}{n} \\ & \frac{1}{2n}, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{p-1} \pmod{p^4}.$$

The last congruence follows from the fact that the rising factorial $(\frac{1}{n})_k$ contains a multiple of p for $\frac{p-1}{n} < k \leq p-1$.

Now, as shown in the proof of Theorem 1.2, we can find $a_{k,1}, a_{k,2}, \dots \in \mathbb{Z}_p$ such that

$$(3.53) \quad \frac{\left(\frac{1}{n} - \frac{x}{n}\right)_k}{\left(1 + \frac{x}{n}\right)_k} = \frac{\left(\frac{1}{n}\right)_k}{(1)_k} \left[1 + \sum_{i \geq 1} a_{k,i} x^i\right]$$

for $0 \leq k \leq \frac{p-1}{n}$. Thus,

$$\begin{aligned}
& {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1-x}{n}, & \frac{1-y}{n}, & \frac{1-p}{n} \\ \frac{1}{2n}, & 1 + \frac{x}{n}, & 1 + \frac{y}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\
& \equiv {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1-x}{n}, & \frac{1-y}{n}, & \frac{1}{n} \\ \frac{1}{2n}, & 1 + \frac{x}{n}, & 1 + \frac{y}{n}, & 1 \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \pmod{p} \\
& = \sum_{k=0}^{\frac{p-1}{n}} \frac{(1 + \frac{1}{2n})_k (\frac{1}{n})_k^4}{(\frac{1}{2n})_k k!^4} \left[1 + \sum_{i \geq 1} a_{k,i} x^i \right] \left[1 + \sum_{i \geq 1} a_{k,i} y^i \right] \in \mathbb{Z}_p[[x, y]].
\end{aligned}$$

Again, Theorem 2.7 implies that

$$\begin{aligned}
& {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1-x}{n}, & \frac{1-y}{n}, & \frac{1-p}{n} \\ \frac{1}{2n}, & 1 + \frac{x}{n}, & 1 + \frac{y}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\
& = \frac{(1 + \frac{1}{n})_{\frac{p-1}{n}} (\frac{n-1}{n} + \frac{(x+y)}{n})_{\frac{p-1}{n}}}{(1 + \frac{x}{n})_{\frac{p-1}{n}} (1 + \frac{y}{n})_{\frac{p-1}{n}}}.
\end{aligned}$$

We note that the rising factorial $(1 + \frac{1}{n})_{\frac{p-1}{n}}$ contains a factor which is a multiple of p , namely $(1 + \frac{1}{n} + \frac{p-1}{n} - 1) = \frac{p}{n}$; and the other factors do not contain any multiple of p . Therefore, we have

$$(3.54) \quad {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1-x}{n}, & \frac{1-y}{n}, & \frac{1-p}{n} \\ \frac{1}{2n}, & 1 + \frac{x}{n}, & 1 + \frac{y}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \in p\mathbb{Z}_p[[x, y]].$$

If we put $x = \zeta u$ and $y = \zeta^2 u$, then using Lemma 2.4 we obtain

$$\begin{aligned}
& {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1-\zeta u}{n}, & \frac{1-\zeta^2 u}{n}, & \frac{1-u}{n} \\ \frac{1}{2n}, & 1 + \frac{\zeta u}{n}, & 1 + \frac{\zeta^2 u}{n}, & 1 + \frac{u}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\
& = A_0 + \sum_{i \geq 1} A_{3i} u^{3i} \in \mathbb{Z}_p[[u^3]],
\end{aligned}$$

where $A_0 = \sum_{k=0}^{\frac{p-1}{n}} \frac{(1 + \frac{1}{2n})_k (\frac{1}{n})_k^4}{(\frac{1}{2n})_k k!^4}$ and $A_3 = \sum_{k=0}^{\frac{p-1}{n}} \frac{(1 + \frac{1}{2n})_k (\frac{1}{n})_k^4}{(\frac{1}{2n})_k k!^4} H(a_{k,i})$ and $H(a_{k,i})$ is an integral polynomial in the $a_{k,i}$ where the second subscripts in each monomial add to 3. If we put $u = 0$ and p , then (3.54) implies that A_0 and $A_3 \in p\mathbb{Z}_p$. Thus,

$$\begin{aligned}
& {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1-\zeta p}{n}, & \frac{1-\zeta^2 p}{n}, & \frac{1-p}{n} \\ \frac{1}{2n}, & 1 + \frac{\zeta p}{n}, & 1 + \frac{\zeta^2 p}{n}, & 1 + \frac{p}{n} \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \\
& \equiv \sum_{k=0}^{\frac{p-1}{n}} \frac{(1 + \frac{1}{2n})_k (\frac{1}{n})_k^4}{(\frac{1}{2n})_k k!^4} \\
(3.55) \quad & \equiv {}_5F_4 \left[\begin{matrix} \frac{1}{n}, & 1 + \frac{1}{2n}, & \frac{1}{n}, & \frac{1}{n}, & \frac{1}{n} \\ \frac{1}{2n}, & 1, & 1, & 1, & 1 \end{matrix} ; 1 \right]_{\frac{p-1}{n}} \pmod{p^4},
\end{aligned}$$

which proves (3.52). Again, applying 2nd part of Proposition 2.1, (2.3) and (2.4) on the right hand side of (3.51), we deduce that

$$(3.56) \quad \frac{(1 + \frac{1}{n})_{\frac{p-1}{n}} (\frac{n-1}{n} + \frac{(\zeta + \zeta^2)p}{n})_{\frac{p-1}{n}}}{(1 + \frac{\zeta p}{n})_{\frac{p-1}{n}} (1 + \frac{\zeta^2 p}{n})_{\frac{p-1}{n}}} = -p \cdot \frac{\Gamma_p(1 + \frac{p}{n}) \Gamma_p(1 + \frac{p\zeta}{n}) \Gamma_p(1 + \frac{p\zeta^2}{n}) \Gamma_p(\frac{n-2}{n})}{\Gamma_p(\frac{1}{n}) \Gamma_p(\frac{n-1}{n} - \frac{p}{n}) \Gamma_p(\frac{n-1}{n} - \frac{p\zeta}{n}) \Gamma_p(\frac{n-1}{n} - \frac{p\zeta^2}{n})}.$$

As shown in the proof of Theorem 1.3, we use Theorem 2.3 on the right side of (3.56) to obtain, for $p \geq 5$

$$(3.57) \quad \frac{(1 + \frac{1}{n})_{\frac{p-1}{n}} (\frac{n-1}{n} + \frac{(\zeta + \zeta^2)p}{n})_{\frac{p-1}{n}}}{(1 + \frac{\zeta p}{n})_{\frac{p-1}{n}} (1 + \frac{\zeta^2 p}{n})_{\frac{p-1}{n}}} \equiv (-1)^{1 + \frac{p-1}{n}} p \Gamma_p \left(\frac{1}{n} \right)^2 \Gamma_p \left(\frac{n-2}{n} \right) \pmod{p^4}.$$

Finally, from (3.51), (3.52), (3.55) and (3.57) we obtain our result. \square

Proof of Theorem 1.5. Let $a = \frac{1}{8}$, $b = \frac{1-\zeta p}{4}$, $c = \frac{1-\zeta^2 p}{4}$, $d = \frac{1-\zeta^3 p}{4}$, $e = \frac{1-\zeta^4 p}{4}$, $f = \frac{1-p}{4}$, where ζ is a primitive 5th root of unity. Substituting all these values in Theorem 2.6 we have

$$(3.58) \quad {}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1-\zeta p}{4}, & \frac{1-\zeta^2 p}{4}, & \frac{1-\zeta^3 p}{4}, & \frac{1-\zeta^4 p}{4}, & \frac{1-p}{4} \\ \frac{1}{16}, & \frac{7}{8} + \frac{\zeta p}{4}, & \frac{7}{8} + \frac{\zeta^2 p}{4}, & \frac{7}{8} + \frac{\zeta^3 p}{4}, & \frac{7}{8} + \frac{\zeta^4 p}{4}, & \frac{7}{8} + \frac{p}{4} \end{matrix} ; 1 \right]_{\frac{p-1}{4}} \\ = \frac{(\frac{9}{8})_{\frac{p-1}{4}} (\frac{5+2\zeta p+2\zeta^2 p}{8})_{\frac{p-1}{4}} (\frac{5+2\zeta p+2\zeta^3 p}{8})_{\frac{p-1}{4}} (\frac{5+2\zeta^2 p+2\zeta^3 p}{8})_{\frac{p-1}{4}}}{(\frac{7}{8} + \frac{\zeta p}{4})_{\frac{p-1}{4}} (\frac{7}{8} + \frac{\zeta^2 p}{4})_{\frac{p-1}{4}} (\frac{7}{8} + \frac{\zeta^3 p}{4})_{\frac{p-1}{4}} (\frac{3+2\zeta p+2\zeta^2 p+2\zeta^3 p}{8})_{\frac{p-1}{4}}}.$$

Our claim is that

$$(3.59) \quad {}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1-\zeta p}{4}, & \frac{1-\zeta^2 p}{4}, & \frac{1-\zeta^3 p}{4}, & \frac{1-\zeta^4 p}{4}, & \frac{1-p}{4} \\ \frac{1}{16}, & \frac{7}{8} + \frac{\zeta p}{4}, & \frac{7}{8} + \frac{\zeta^2 p}{4}, & \frac{7}{8} + \frac{\zeta^3 p}{4}, & \frac{7}{8} + \frac{\zeta^4 p}{4}, & \frac{7}{8} + \frac{p}{4} \end{matrix} ; 1 \right]_{\frac{p-1}{4}} \\ \equiv {}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4} \\ \frac{1}{16}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8} \end{matrix} ; 1 \right]_{\frac{p-1}{4}} \\ \equiv {}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4} \\ \frac{1}{16}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8} \end{matrix} ; 1 \right]_{p-1} \pmod{p^6}.$$

The last congruence follows easily from the fact that the rising factorials $(\frac{1}{8})_k$ and $(\frac{1}{4})_k$ contain a multiple of p for each k satisfying $\frac{p-1}{4} < k \leq p-1$. Again,

$$(3.60) \quad \frac{(\frac{1}{4} - x)_k}{(\frac{7}{8} + x)_k} = \prod_{j=0}^{k-1} \frac{(\frac{1}{4} + j - x)}{(\frac{7}{8} + j + x)} = \prod_{j=0}^{k-1} \frac{(\frac{1}{4} + j)(1 - \frac{4x}{1+4j})}{(\frac{7}{8} + j)(1 + \frac{8x}{7+8j})} \\ = \frac{(\frac{1}{4})_k}{(\frac{7}{8})_k} [1 + a_{k,1}x + a_{k,2}x^2 + \dots]$$

for some constants $a_{k,1}, a_{k,2}, \dots$. We observe that the terms $\frac{1}{4} + j$ and $\frac{7}{8} + j$ do not contain a multiple of p for each k in the range $0 \leq k \leq \frac{p-1}{4}$, and hence

$a_{k,1}, a_{k,2}, \dots \in \mathbb{Z}_p$. Now, (3.60) yields

$$\begin{aligned}
& {}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1}{4} - x, & \frac{1}{4} - y, & \frac{1}{4} - z, & \frac{1}{4} - w, & \frac{1-p}{4} \\ \frac{1}{16}, & \frac{1}{16}, & \frac{7}{8} + x, & \frac{7}{8} + y, & \frac{7}{8} + z, & \frac{7}{8} + w, & \frac{7}{8} + \frac{p}{4} \end{matrix} ; 1 \right]_{\frac{p-1}{4}} \\
& \equiv {}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1}{4} - x, & \frac{1}{4} - y, & \frac{1}{4} - z, & \frac{1}{4} - w, & \frac{1}{4} \\ \frac{1}{16}, & \frac{1}{16}, & \frac{7}{8} + x, & \frac{7}{8} + y, & \frac{7}{8} + z, & \frac{7}{8} + w, & \frac{7}{8} \end{matrix} ; 1 \right]_{\frac{p-1}{4}} \pmod{p} \\
& = \sum_{k=0}^{\frac{p-1}{4}} \frac{(16k+1)(\frac{1}{8})_k(\frac{1}{4})_k^5}{(\frac{7}{8})_k^5 k!} \left[1 + \sum_{i \geq 1} a_{k,i} x^i \right] \left[1 + \sum_{i \geq 1} a_{k,i} y^i \right] \left[1 + \sum_{i \geq 1} a_{k,i} z^i \right] \\
& \times \left[1 + \sum_{i \geq 1} a_{k,i} w^i \right] \in \mathbb{Z}_p[[x, y, z, w]].
\end{aligned} \tag{3.61}$$

Using Theorem 2.5 we obtain

$$\begin{aligned}
& {}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1}{4} - x, & \frac{1}{4} - y, & \frac{1}{4} - z, & \frac{1}{4} - w, & \frac{1-p}{4} \\ \frac{1}{16}, & \frac{1}{16}, & \frac{7}{8} + x, & \frac{7}{8} + y, & \frac{7}{8} + z, & \frac{7}{8} + w, & \frac{7}{8} + \frac{p}{4} \end{matrix} ; 1 \right]_{\frac{p-1}{4}} \\
& = \frac{(\frac{9}{8})_{\frac{p-1}{4}} (\frac{5}{8} + z + w)_{\frac{p-1}{4}}}{(\frac{7}{8} + z)_{\frac{p-1}{4}} (\frac{7}{8} + w)_{\frac{p-1}{4}}} \\
& \times {}_4F_3 \left[\begin{matrix} \frac{5}{8} + x + y, & \frac{1}{4} - z, & \frac{1}{4} - w, \\ \frac{7}{8} + x, & \frac{7}{8} + y, & \frac{5}{8} - z - w - \frac{p}{4} \end{matrix} ; 1 \right]_{\frac{p-1}{4}}.
\end{aligned}$$

The rising factorial $(\frac{9}{8})_{\frac{p-1}{4}}$ contains a multiple of p , namely $(\frac{9}{8} + \frac{p-1}{8} - 1) = \frac{p}{8}$; and the terms in the denominator are units. Therefore, we have

$${}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1}{4} - x, & \frac{1}{4} - y, & \frac{1}{4} - z, & \frac{1}{4} - w, & \frac{1-p}{4} \\ \frac{1}{16}, & \frac{1}{16}, & \frac{7}{8} + x, & \frac{7}{8} + y, & \frac{7}{8} + z, & \frac{7}{8} + w, & \frac{7}{8} + \frac{p}{4} \end{matrix} ; 1 \right]_{\frac{p-1}{4}} \in p\mathbb{Z}_p[[x, y, z, w]]. \tag{3.62}$$

Let $x = \zeta u, y = \zeta^2 u, z = \zeta^3 u$ and $w = \zeta^4 u$, where u is any p -adic integer. Then (3.61) and Lemma 2.4 yield

$$\begin{aligned}
& {}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1}{4} - \zeta u, & \frac{1}{4} - \zeta^2 u, & \frac{1}{4} - \zeta^3 u, & \frac{1}{4} - \zeta^4 u, & \frac{1}{4} - u \\ \frac{1}{16}, & \frac{1}{16}, & \frac{7}{8} + \zeta u, & \frac{7}{8} + \zeta^2 u, & \frac{7}{8} + \zeta^3 u, & \frac{7}{8} + \zeta^4 u, & \frac{7}{8} + u \end{matrix} ; 1 \right]_{\frac{p-1}{4}} \\
& = A_0 + \sum_{i \geq 1} A_{5i} u^{5i} \in \mathbb{Z}_p[[u^5]],
\end{aligned}$$

where $A_0 = \sum_{k=0}^{\frac{p-1}{4}} \frac{(16k+1)(\frac{1}{8})_k(\frac{1}{4})_k^5}{(\frac{7}{8})_k^5 k!}$ and $A_5 = \sum_{k=0}^{\frac{p-1}{4}} \frac{(16k+1)(\frac{1}{8})_k(\frac{1}{4})_k^5}{(\frac{7}{8})_k^5 k!} H(a_{k,i})$; and $H(a_{k,i})$ is an integral polynomial in the $a_{k,i}$ where the second subscripts in each monomial add to 5. If we put $u = 0$ and $\frac{p}{4}$, then (3.62) implies that A_0 and

$A_5 \in p\mathbb{Z}_p$. Therefore, we conclude that

$$\begin{aligned}
 & {}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1-\zeta p}{4}, & \frac{1-\zeta^2 p}{4}, & \frac{1-\zeta^3 p}{4}, & \frac{1-\zeta^4 p}{4}, & \frac{1-p}{4} \\ \frac{1}{16}, & \frac{7}{8} + \frac{\zeta p}{4}, & \frac{7}{8} + \frac{\zeta^2 p}{4}, & \frac{7}{8} + \frac{\zeta^3 p}{4}, & \frac{7}{8} + \frac{\zeta^4 p}{4}, & \frac{7}{8} + \frac{p}{4} \end{matrix} ; 1 \right]_{\frac{p-1}{4}} \\
 & \equiv \sum_{k=0}^{\frac{p-1}{4}} \frac{(16k+1)(\frac{1}{8})_k(\frac{1}{4})_k^5}{(\frac{7}{8})_k^5 k!} \\
 (3.63) \quad & = {}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4} \\ \frac{1}{16}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8} \end{matrix} ; 1 \right]_{\frac{p-1}{4}} \pmod{p^6}.
 \end{aligned}$$

This proves our claim (3.59).

If we apply (2.3) and (2.4), and then 2nd part of Proposition 2.1 on the right hand side of (3.58) as in Theorem 1.2, we deduce that

$$\begin{aligned}
 & \frac{(\frac{9}{8})_{\frac{p-1}{4}} (\frac{5+2\zeta p+2\zeta^2 p}{8})_{\frac{p-1}{4}} (\frac{5+2\zeta p+2\zeta^3 p}{8})_{\frac{p-1}{4}} (\frac{5+2\zeta^2 p+2\zeta^3 p}{8})_{\frac{p-1}{4}}}{(\frac{7}{8} + \frac{\zeta p}{4})_{\frac{p-1}{4}} (\frac{7}{8} + \frac{\zeta^2 p}{4})_{\frac{p-1}{4}} (\frac{7}{8} + \frac{\zeta^3 p}{4})_{\frac{p-1}{4}} (\frac{3+2\zeta p+2\zeta^2 p+2\zeta^3 p}{8})_{\frac{p-1}{4}}} \\
 & = \frac{-p\Gamma_p(\frac{7+2p}{8})\Gamma_p(\frac{3+2p+2\zeta p+2\zeta^2 p}{8})\Gamma_p(\frac{3+2p+2\zeta p+2\zeta^3 p}{8})\Gamma_p(\frac{3+2p+2\zeta^2 p+2\zeta^3 p}{8})}{\Gamma_p(\frac{1}{8})\Gamma_p(\frac{5+2\zeta p+2\zeta^2 p}{8})\Gamma_p(\frac{5+2\zeta p+2\zeta^3 p}{8})\Gamma_p(\frac{5+2\zeta^2 p+2\zeta^3 p}{8})} \\
 (3.64) \quad & \times \frac{\Gamma_p(\frac{7}{8} + \frac{\zeta p}{4})\Gamma_p(\frac{7}{8} + \frac{\zeta^2 p}{4})\Gamma_p(\frac{7}{8} + \frac{\zeta^3 p}{4})\Gamma_p(\frac{3+2\zeta p+2\zeta^2 p+2\zeta^3 p}{8})}{\Gamma_p(\frac{5}{8} + \frac{(1+\zeta)p}{4})\Gamma_p(\frac{5}{8} + \frac{(1+\zeta^2)p}{4})\Gamma_p(\frac{5}{8} + \frac{(1+\zeta^3)p}{4})\Gamma_p(\frac{1+2p+2\zeta p+2\zeta^2 p+2\zeta^3 p}{8})}.
 \end{aligned}$$

Rearranging the terms on the right side of (3.64) with respect to the symmetry of the 5th root of unity, we obtain

$$\begin{aligned}
 & \frac{(\frac{9}{8})_{\frac{p-1}{4}} (\frac{5+2\zeta p+2\zeta^2 p}{8})_{\frac{p-1}{4}} (\frac{5+2\zeta p+2\zeta^3 p}{8})_{\frac{p-1}{4}} (\frac{5+2\zeta^2 p+2\zeta^3 p}{8})_{\frac{p-1}{4}}}{(\frac{7}{8} + \frac{\zeta p}{4})_{\frac{p-1}{4}} (\frac{7}{8} + \frac{\zeta^2 p}{4})_{\frac{p-1}{4}} (\frac{7}{8} + \frac{\zeta^3 p}{4})_{\frac{p-1}{4}} (\frac{3+2\zeta p+2\zeta^2 p+2\zeta^3 p}{8})_{\frac{p-1}{4}}} \\
 & = \frac{-p\Gamma_p(\frac{7+2p}{8})\Gamma_p(\frac{7}{8} + \frac{\zeta p}{4})\Gamma_p(\frac{7}{8} + \frac{\zeta^2 p}{4})\Gamma_p(\frac{7}{8} + \frac{\zeta^3 p}{4})}{\Gamma_p(\frac{1}{8})\Gamma_p(\frac{1+2p+2\zeta p+2\zeta^2 p+2\zeta^3 p}{8})} \\
 & \times \frac{\Gamma_p(\frac{3+2p+2\zeta p+2\zeta^2 p}{8})\Gamma_p(\frac{3+2p+2\zeta p+2\zeta^3 p}{8})\Gamma_p(\frac{3+2p+2\zeta^2 p+2\zeta^3 p}{8})}{\Gamma_p(\frac{5+2\zeta p+2\zeta^2 p}{8})\Gamma_p(\frac{5+2\zeta p+2\zeta^3 p}{8})\Gamma_p(\frac{5+2\zeta^2 p+2\zeta^3 p}{8})} \\
 (3.65) \quad & \times \frac{\Gamma_p(\frac{3+2\zeta p+2\zeta^2 p+2\zeta^3 p}{8})}{\Gamma_p(\frac{5}{8} + \frac{(1+\zeta)p}{4})\Gamma_p(\frac{5}{8} + \frac{(1+\zeta^2)p}{4})\Gamma_p(\frac{5}{8} + \frac{(1+\zeta^3)p}{4})}.
 \end{aligned}$$

Now, using 2nd part of Proposition 2.2 and Theorem 2.3 on the right side of (3.65), we deduce that

$$\begin{aligned}
 (3.66) \quad & \frac{\Gamma_p(\frac{7+2p}{8})\Gamma_p(\frac{7}{8} + \frac{\zeta p}{4})\Gamma_p(\frac{7}{8} + \frac{\zeta^2 p}{4})\Gamma_p(\frac{7}{8} + \frac{\zeta^3 p}{4})}{\Gamma_p(\frac{1+2p+2\zeta p+2\zeta^2 p+2\zeta^3 p}{8})} = (-1)^{1+\frac{p-1}{8}} \Gamma_p \left(\frac{7}{8} \right)^5 [1 + O(p^5)],
 \end{aligned}$$

and

$$\begin{aligned}
& \frac{\Gamma_p(\frac{3}{8} + \frac{(1+\zeta+\zeta^2)p}{4})\Gamma_p(\frac{3}{8} + \frac{(1+\zeta+\zeta^3)p}{4})\Gamma_p(\frac{3}{8} + \frac{(1+\zeta^2+\zeta^3)p}{4})}{\Gamma_p(\frac{5}{8} + \frac{(\zeta+\zeta^2)p}{4})\Gamma_p(\frac{5}{8} + \frac{(\zeta+\zeta^3)p}{4})\Gamma_p(\frac{5}{8} + \frac{(\zeta^2+\zeta^3)p}{4})\Gamma_p(\frac{5}{8} + \frac{(1+\zeta)p}{4})} \\
& \times \frac{\Gamma_p(\frac{3}{8} + \frac{(\zeta+\zeta^2+\zeta^3)p}{4})}{\Gamma_p(\frac{5}{8} + \frac{(1+\zeta^2)p}{4})\Gamma_p(\frac{5}{8} + \frac{(1+\zeta^3)p}{4})} \\
(3.67) \quad & = \Gamma_p\left(\frac{3}{8}\right)^{10} [1 + O(p^5)].
\end{aligned}$$

Again, using (3.66), (3.67) and $\Gamma_p\left(\frac{1}{8}\right)\Gamma_p\left(\frac{7}{8}\right) = (-1)^{1+\frac{p-1}{8}}$, we obtain

$$\begin{aligned}
& \frac{\left(\frac{9}{8}\right)^{\frac{p-1}{4}}\left(\frac{5+2\zeta p+2\zeta^2 p}{8}\right)^{\frac{p-1}{4}}\left(\frac{5+2\zeta p+2\zeta^3 p}{8}\right)^{\frac{p-1}{4}}\left(\frac{5+2\zeta^2 p+2\zeta^3 p}{8}\right)^{\frac{p-1}{4}}}{\left(\frac{7}{8} + \frac{\zeta p}{4}\right)^{\frac{p-1}{4}}\left(\frac{7}{8} + \frac{\zeta^2 p}{4}\right)^{\frac{p-1}{4}}\left(\frac{7}{8} + \frac{\zeta^3 p}{4}\right)^{\frac{p-1}{4}}\left(\frac{3+2\zeta p+2\zeta^2 p+2\zeta^3 p}{8}\right)^{\frac{p-1}{4}}} \\
(3.68) \quad & = -p\Gamma_p\left(\frac{7}{8}\right)^6 \Gamma_p\left(\frac{3}{8}\right)^{10} (1 + O(p^5)).
\end{aligned}$$

Finally, from (3.58), (3.63), (3.59) and (3.68) we obtain

$${}_7F_6 \left[\begin{matrix} \frac{1}{8}, & \frac{17}{16}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4} \\ \frac{1}{16}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8}, & \frac{7}{8} \end{matrix} ; 1 \right]_{p-1} \equiv -p\Gamma_p\left(\frac{7}{8}\right)^6 \Gamma_p\left(\frac{3}{8}\right)^{10} \pmod{p^6}.$$

This completes the proof of the theorem. \square

Proof of Theorem 1.6. Let $m_1 = m_2 = \dots = m_n = \frac{p-1}{n(n-1)}$, $b_1 = \frac{1}{n-1} + \frac{yp}{n-1}$ and $b_2 = \dots = b_n = \frac{1}{n-1}$, $y \in \mathbb{Z}_p$. Substituting these values in Theorem 2.8 we have

$$\begin{aligned}
& {}_{n+1}F_n \left[\begin{matrix} \frac{1-p}{n-1}, & \frac{1}{n} + \frac{p}{n-1}(y + \frac{1}{n}), & \frac{1}{n} + \frac{p}{n(n-1)}, & \dots, & \frac{1}{n} + \frac{p}{n(n-1)} \\ \frac{1}{n-1} + \frac{yp}{n-1}, & \frac{1}{n-1}, & \dots, & \frac{1}{n-1} \end{matrix} ; 1 \right]_{\frac{p-1}{n-1}} \\
(3.69) \quad & = \frac{(-1)^{\frac{p-1}{n-1}} \left(\frac{p-1}{n-1}\right)!}{\left(\frac{1}{n-1} + \frac{yp}{n-1}\right)^{\frac{p-1}{n(n-1)}} \left(\frac{1}{n-1}\right)^{\frac{p-1}{n(n-1)}} \dots \left(\frac{1}{n-1}\right)^{\frac{p-1}{n(n-1)}}}.
\end{aligned}$$

By definition we have

$$\begin{aligned}
& {}_{n+1}F_n \left[\begin{matrix} \frac{1-p}{n-1}, & \frac{1}{n} + \frac{p}{n-1}(y + \frac{1}{n}), & \frac{1}{n} + \frac{p}{n(n-1)}, & \dots, & \frac{1}{n} + \frac{p}{n(n-1)} \\ \frac{1}{n-1} + \frac{yp}{n-1}, & \frac{1}{n-1}, & \dots, & \frac{1}{n-1} \end{matrix} ; 1 \right]_{\frac{p-1}{n-1}} \\
(3.70) \quad & = \sum_{k=0}^{\frac{p-1}{n-1}} \frac{\left(\frac{1-p}{n-1}\right)_k \left(\frac{1}{n} + \frac{p}{n-1}(y + \frac{1}{n})\right)_k \left(\frac{1}{n} + \frac{p}{n(n-1)}\right)_k^{n-1}}{\left(\frac{1}{n-1} + \frac{yp}{n-1}\right)_k \left(\frac{1}{n-1}\right)_k^{n-1}} \frac{1}{k!}.
\end{aligned}$$

Now,

$$\begin{aligned}
\left(\frac{1-p}{n-1}\right)_k & = \frac{1-p}{n-1} \left(\frac{1-p}{n-1} + 1\right) \dots \left(\frac{1-p}{n-1} + k-1\right) \\
& \equiv \left(\frac{1}{n-1}\right)_k \left(1 - p \sum_{i=0}^{k-1} \frac{1}{(n-1)\left(\frac{1}{n-1} + i\right)}\right) \pmod{p^2}.
\end{aligned}$$

Similarly, we have

$$\left(\frac{1}{n} + \frac{p}{n-1} \left(y + \frac{1}{n}\right)\right)_k \equiv \left(\frac{1}{n}\right)_k \left(1 + \frac{p(\frac{1}{n} + y)}{n-1} \sum_{i=0}^{k-1} \frac{1}{\frac{1}{n} + i}\right) \pmod{p^2},$$

$$\left(\frac{1}{n} + \frac{p}{n(n-1)}\right)_k \equiv \left(\frac{1}{n}\right)_k \left(1 + \frac{p}{n(n-1)} \sum_{i=0}^{k-1} \frac{1}{\frac{1}{n} + i}\right) \pmod{p^2}$$

and

$$\left(\frac{1}{n-1} + \frac{yp}{n-1}\right)_k \equiv \left(\frac{1}{n-1}\right)_k \left(1 + yp \sum_{i=0}^{k-1} \frac{1}{(n-1)(\frac{1}{n-1} + i)}\right) \pmod{p^2}.$$

Also,

$$\begin{aligned} \frac{(\frac{1-p}{n-1})_k}{(\frac{1}{n-1} + \frac{yp}{n-1})_k} &\equiv \frac{(\frac{1}{n-1})_k \left(1 - p \sum_{i=0}^{k-1} \frac{1}{(n-1)(\frac{1}{n-1} + i)}\right)}{(\frac{1}{n-1})_k \left(1 + yp \sum_{i=0}^{k-1} \frac{1}{(n-1)(\frac{1}{n-1} + i)}\right)} \pmod{p^2} \\ &\equiv 1 - (1+y)p \sum_{i=0}^{k-1} \frac{1}{(n-1)(\frac{1}{n-1} + i)} \pmod{p^2} \\ &\equiv 1 + (1+y)A_k p \pmod{p^2} \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{1}{n} + \frac{p}{n-1} \left(y + \frac{1}{n}\right)\right)_k \left(\frac{1}{n} + \frac{p}{n(n-1)}\right)_k^{n-1} \\ &\equiv \left(\frac{1}{n}\right)_k \left(1 + \frac{p(\frac{1}{n} + y)}{n-1} \sum_{i=0}^{k-1} \frac{1}{\frac{1}{n} + i}\right) \left(\frac{1}{n}\right)_k^{n-1} \left(1 + \frac{(n-1)p}{n(n-1)} \sum_{i=0}^{k-1} \frac{1}{\frac{1}{n} + i}\right) \pmod{p^2} \\ &\equiv \left(\frac{1}{n}\right)_k^n \left(1 + \frac{p(1+y)}{n-1} \sum_{i=0}^{k-1} \frac{1}{\frac{1}{n} + i}\right) \pmod{p^2} \\ &\equiv \left(\frac{1}{n}\right)_k^n (1 + p(1+y)B_k) \pmod{p^2}. \end{aligned}$$

Here, $A_k = -\sum_{i=0}^{k-1} \frac{1}{(n-1)(\frac{1}{n-1} + i)} \in \mathbb{Z}_p$ and $B_k = \frac{1}{n-1} \sum_{i=0}^{k-1} \frac{1}{\frac{1}{n} + i} \in \mathbb{Z}_p$. Using all these identities on the right hand side of (3.70) we find that

$$\begin{aligned} &_{n+1}F_n \left[\begin{matrix} \frac{1-p}{n-1}, & \frac{1}{n} + \frac{p}{n-1} \left(y + \frac{1}{n}\right), & \frac{1}{n} + \frac{p}{n(n-1)}, & \dots, & \frac{1}{n} + \frac{p}{n(n-1)} \\ & \frac{1}{n-1} + \frac{yp}{n-1}, & \frac{1}{n-1}, & \dots, & \frac{1}{n-1} \end{matrix} ; 1 \right]_{\frac{p-1}{n-1}} \\ &\equiv \sum_{k=0}^{\frac{p-1}{n-1}} \frac{(\frac{1}{n})_k^n}{(\frac{1}{n-1})_k^{n-1} k!} \{1 + (1+y)(A_k + B_k)p\} \pmod{p^2} \\ (3.71) \quad &\equiv {}_nF_{n-1} \left[\begin{matrix} \frac{1}{n}, & \frac{1}{n-1}, & \dots, & \frac{1}{n-1} \\ & \frac{1}{n-1}, & \dots, & \frac{1}{n-1} \end{matrix} ; 1 \right]_{\frac{p-1}{n-1}} + (1+y)Cp \pmod{p^2}, \end{aligned}$$

for some $C \in \mathbb{Z}_p$. From (2.3), we have $\left(\frac{p-1}{n-1}\right)! = (1)_{\frac{p-1}{n-1}} = -\Gamma_p\left(1 + \frac{p-1}{n-1}\right)$; and then using Proposition 2.1 we deduce that $\left(\frac{p-1}{n-1}\right)! = \frac{1}{\Gamma_p(\frac{1-p}{n-1})}$. Again, (2.3) yields

$$\left(\frac{1}{n-1} + \frac{yp}{n-1}\right)_{\frac{p-1}{n(n-1)}} = (-1)^{\frac{p-1}{n(n-1)}} \frac{\Gamma_p\left(\frac{1}{n-1} + \frac{yp}{n-1} + \frac{p-1}{n(n-1)}\right)}{\Gamma_p\left(\frac{1}{n-1} + \frac{yp}{n-1}\right)}$$

and

$$\left(\frac{1}{n-1}\right)_{\frac{p-1}{n(n-1)}} = (-1)^{\frac{p-1}{n(n-1)}} \frac{\Gamma_p\left(\frac{1}{n-1} + \frac{p-1}{n(n-1)}\right)}{\Gamma_p\left(\frac{1}{n-1}\right)}.$$

Putting these values in (3.69), we find that the right hand side reduces to

$$\begin{aligned} & \frac{(-1)^{\frac{p-1}{n-1} + \frac{p-1}{n} + \frac{p-1}{n(n-1)}} \Gamma_p\left(\frac{1}{n-1} + \frac{yp}{n-1}\right) \Gamma_p\left(\frac{1}{n-1}\right)^{n-1}}{\Gamma_p\left(\frac{1-p}{n-1}\right) \Gamma_p\left(\frac{1}{n-1} + \frac{yp}{n-1} + \frac{p-1}{n(n-1)}\right) \Gamma_p\left(\frac{1}{n-1} + \frac{p-1}{n(n-1)}\right)^{n-1}} \\ (3.72) \quad &= \frac{\Gamma_p\left(\frac{1}{n-1} + \frac{yp}{n-1}\right) \Gamma_p\left(\frac{1}{n-1}\right)^{n-1}}{\Gamma_p\left(\frac{1}{n-1} - \frac{p}{n-1}\right) \Gamma_p\left(\frac{1}{n} + \frac{(ny+1)p}{n(n-1)}\right) \Gamma_p\left(\frac{1}{n} + \frac{p}{n(n-1)}\right)^{n-1}}. \end{aligned}$$

From Theorem 2.3 we have, for $p \geq 5$

$$\begin{aligned} \frac{\Gamma_p\left(\frac{1}{n-1} + \frac{yp}{n-1}\right)}{\Gamma_p\left(\frac{1}{n-1} - \frac{p}{n-1}\right)} &\equiv \frac{\Gamma_p\left(\frac{1}{n-1}\right) \left(1 + G_1\left(\frac{1}{n-1}\right) \frac{yp}{n-1}\right)}{\Gamma_p\left(\frac{1}{n-1}\right) \left(1 - G_1\left(\frac{1}{n-1}\right) \frac{p}{n-1}\right)} \\ &\equiv \left(1 + G_1\left(\frac{1}{n-1}\right) \frac{yp}{n-1}\right) \left(1 + G_1\left(\frac{1}{n-1}\right) \frac{p}{n-1}\right) \\ (3.73) \quad &\equiv \left(1 + G_1\left(\frac{1}{n-1}\right) \frac{(1+y)p}{n-1}\right) \pmod{p^2} \end{aligned}$$

and

$$\begin{aligned} & \Gamma_p\left(\frac{1}{n} + \frac{(ny+1)p}{n(n-1)}\right) \Gamma_p\left(\frac{1}{n} + \frac{p}{n(n-1)}\right)^{n-1} \\ & \equiv \Gamma_p\left(\frac{1}{n}\right) \left(1 + G_1\left(\frac{1}{n}\right) \frac{(ny+1)p}{n(n-1)}\right) \Gamma_p\left(\frac{1}{n}\right)^{n-1} \left(1 + G_1\left(\frac{1}{n}\right) \frac{p}{n(n-1)}\right)^{n-1} \\ & \equiv \Gamma_p\left(\frac{1}{n}\right)^n \left(1 + G_1\left(\frac{1}{n}\right) \frac{(ny+1)p}{n(n-1)}\right) \left(1 + G_1\left(\frac{1}{n}\right) \frac{p}{n}\right) \\ (3.74) \quad & \equiv \Gamma_p\left(\frac{1}{n}\right)^n \left(1 + G_1\left(\frac{1}{n}\right) \frac{(1+y)p}{n-1}\right) \pmod{p^2}. \end{aligned}$$

If we use (3.72), (3.73) and (3.74), then modulo p^2 the right hand side of (3.69) becomes

$$(3.75) \quad \frac{\Gamma_p\left(\frac{1}{n-1}\right)^{n-1}}{\Gamma_p\left(\frac{1}{n}\right)^n} \left[1 + \left(G_1\left(\frac{1}{n-1}\right) - G_1\left(\frac{1}{n}\right)\right) \frac{(1+y)p}{n-1}\right].$$

Now, from (3.69), (3.71), (3.75) and Proposition 2.1 we deduce that, modulo p^2

$$\begin{aligned}
 & {}_nF_{n-1} \left[\begin{matrix} \frac{1}{n}, & \frac{1}{n}, & \dots, & \frac{1}{n} \\ \frac{1}{n-1}, & \dots, & \frac{1}{n-1} \end{matrix} ; 1 \right]_{\frac{p-1}{n-1}} + (1+y)Cp \\
 & \equiv \frac{\Gamma_p(\frac{1}{n-1})^{n-1}}{\Gamma_p(\frac{1}{n})^n} \left[1 + \left(G_1 \left(\frac{1}{n-1} \right) - G_1 \left(\frac{1}{n} \right) \right) \frac{(1+y)p}{n-1} \right] \\
 & (3.76) \\
 & \equiv (-1)^n \Gamma_p \left(\frac{1}{n-1} \right)^{n-1} \Gamma_p \left(\frac{n-1}{n} \right)^n \left[1 + \left(G_1 \left(\frac{1}{n-1} \right) - G_1 \left(\frac{1}{n} \right) \right) \frac{(1+y)p}{n-1} \right],
 \end{aligned}$$

which is true for all y . Putting $y = -1$ in (3.76) we obtain the result. We have verified the case $p = 3$ by hand. This completes the proof of the theorem. \square

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